



Homological stability for Artin monoids

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ABSTRACT

We prove that certain sequences of Artin monoids containing the braid monoid as a submonoid satisfy homological stability. When the $K(\pi, 1)$ conjecture holds for the associated family of Artin groups, this establishes homological stability for these groups. In particular, this recovers and extends Arnol’d’s proof of stability for the Artin groups of type A , B and D .

1. Introduction

A sequence of groups or monoids with maps between them

$$G_1 \rightarrow G_2 \rightarrow \cdots \rightarrow G_n \rightarrow \cdots$$

is said to satisfy *homological stability* if the induced maps on homology

$$H_i(G_n) \rightarrow H_i(G_{n+1})$$

are isomorphisms for n sufficiently large compared to i .

This paper concerns homological stability for sequences of *Artin monoids and groups*, and in this paper the associated maps will always be inclusions. In particular, we consider sequences of Artin groups that have the braid group as a subgroup, and the corresponding sequences of monoids.

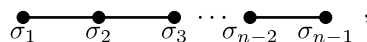
We recall the definition of Artin groups. Given a finite set Σ , to every unordered pair $\{\sigma_s, \sigma_t\} \in \Sigma \times \Sigma$ associate either a natural number greater than 2 or the symbol ∞ , and denote this by $m(s, t)$. An Artin group A with generating set Σ has the following presentation

$$A = \langle \Sigma \mid \pi(\sigma_s, \sigma_t; m(s, t)) = \pi(\sigma_t, \sigma_s; m(s, t)) \rangle,$$

where $\pi(\sigma_s, \sigma_t; m(s, t))$ is the alternating product of σ_s and σ_t starting with σ_s and of length $m(s, t)$. The braid group with its standard presentation is the archetypal example of an Artin group, with presentation

$$\mathcal{B}_n = \left\langle \sigma_i \text{ for } 1 \leq i \leq n-1 \mid \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad 1 \leq i \leq n-2 \end{array} \right\rangle.$$

Every Artin group has an associated *Coxeter group* (discussed in Section 2), and in fact Artin groups were first introduced by Brieskorn [2] as the fundamental groups of hyperplane complements built from Coxeter groups. The information of the presentation can be packaged into a *Coxeter diagram*. This diagram has vertex set Σ and edges corresponding to $m(s, t)$ for each pair of vertices: no edge when $m(s, t) = 2$, an unlabelled edge when $m(s, t) = 3$ and an edge labelled with $m(s, t)$ otherwise. For example, the braid group \mathcal{B}_n has diagram



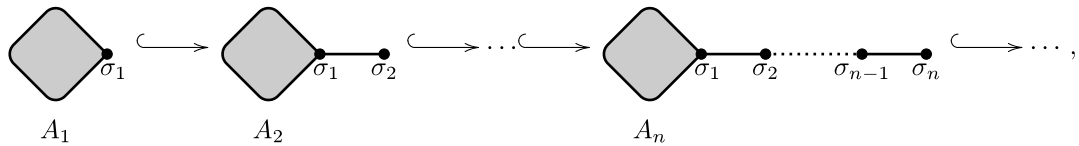
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which is known as the Coxeter diagram of type A and has corresponding Coxeter group the symmetric group S_n .

The sequences of Artin groups studied in this paper correspond to the following sequence of diagrams:



where the grey box indicates that the sequence begins with an arbitrary diagram: an arbitrary Artin group with finite generating set. The type A subdiagram corresponds to a subgroup of A_n being the braid group \mathcal{B}_{n+1} , with an increasing number of generators as n increases. This gives rise to a sequence of groups and inclusions

$$A_1 \hookrightarrow A_2 \hookrightarrow \dots \hookrightarrow A_n \hookrightarrow \dots$$

and the goal of this paper is to discuss stability for sequences of Artin groups of this form. This was motivated by work of Hepworth [13], who proved homological stability for the associated sequence of Coxeter groups.

While the argument used for the proof of homological stability is similar to that used by several authors, the novel part of this paper comes from dealing with Artin groups and monoids. Very little is known for Artin groups in general, for instance, the centre of a generic Artin group is unknown and it is not known whether all Artin groups are torsion free. In particular, there are no tools to date for working with Artin cosets (for example, there is no canonical way to choose a coset representative), something that is usually desirable when proving homological stability for a family of groups. Therefore, the results of this paper are stated and proved for the corresponding *Artin monoids*, for which a technical ‘coset’ theory is developed in Section 4 (the notion of coset of a submonoid is not defined in general). Key properties of Artin monoids, such as the existence of a well-defined length function and the existence of lowest common multiples under certain conditions, allow us to define a canonical choice of ‘coset representative’ for these monoids. From the monoid result, we then deduce homological stability for Artin groups that satisfy the $K(\pi, 1)$ conjecture (discussed in more detail below).

We denote the Artin monoid corresponding to A_n by A_n^+ . The inclusion map between the monoids is denoted s and called the *stabilisation map*. The main result of this paper is the following, which to my knowledge is the first instance where homological stability is proved for monoids.

THEOREM A. *The sequence of Artin monoids*

$$A_1^+ \hookrightarrow A_2^+ \hookrightarrow \dots \hookrightarrow A_n^+ \hookrightarrow \dots$$

satisfies homological stability. More precisely, the induced map on homology

$$H_*(BA_{n-1}^+) \xrightarrow{s_*} H_*(BA_n^+)$$

is an isomorphism when $ < \frac{n}{2}$ and a surjection when $* = \frac{n}{2}$. Here, homology is taken with arbitrary constant coefficients, that is, coefficients in an abelian group.*

The classifying space of an Artin monoid BA^+ is homotopy equivalent to some interesting spaces that arise naturally in mathematics. One manifestation of this is that in the study of Artin groups, there is a well-known conjecture by Arnol’d, Brieskorn, Pham and Thom called the $K(\pi, 1)$ conjecture (discussed in Section 3). For this introduction, it suffices to know the following fact due to Dobrinskaya.

THEOREM [10, Theorem 6.3]. *Given an Artin group A and its associated monoid A^+ , the $K(\pi, 1)$ conjecture holds if and only if the induced map between their classifying spaces $BA^+ \rightarrow BA$ is a homotopy equivalence.*

Thus, if the $K(\pi, 1)$ conjecture holds for a family of Artin groups, Theorem A establishes homological stability as below.

COROLLARY B. *When the $K(\pi, 1)$ conjecture holds for all A_n , the sequence of Artin groups*

$$A_1 \hookrightarrow A_2 \hookrightarrow \dots \hookrightarrow A_n \hookrightarrow \dots$$

satisfies homological stability. More precisely, the induced map on homology

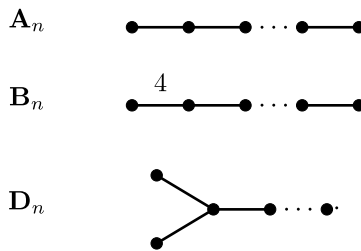
$$H_*(BA_{n-1}) \rightarrow H_*(BA_n)$$

is an isomorphism when $ < \frac{n}{2}$ and a surjection when $* = \frac{n}{2}$. Here, homology is taken with arbitrary constant coefficients, that is, coefficients in an abelian group.*

The $K(\pi, 1)$ conjecture has been proven for large classes of Artin groups [19]: the conjecture holds for Artin groups for which the corresponding Coxeter groups are *finite* (this is *Deligne’s theorem* [9], Theorem 3.1), of *large type*, of *dimension two* and of *FC type*. Proving the $K(\pi, 1)$ conjecture for families of Artin groups continues to be an active area of research to this day.

COROLLARY C. *Homological stability holds for the sequences of Artin groups (A_n) for which the corresponding Coxeter groups are either finite, of large type, of dimension two or of FC type.*

REMARK 1.1. In the case of finite Coxeter groups, Corollary C therefore recovers the few known cases of stability for families of Artin groups of the form studied in this paper, that is, homological stability holds for the sequences of Artin groups $\{A_n\}_{n \geq 1}$ of type *A*, *B* and *D*, given by the following diagrams:



These three sequences consist of Artin groups which relate to finite Coxeter groups. Hence, by Corollary C, the sequences of Artin groups satisfy homological stability.

The three examples in Remark 1.1 were proved by Arnol’d, who computed the full (co)homology of the groups in question, using the associated hyperplane complement. The results and proofs are in Brieskorn’s Bourbaki seminar [3]. Despite the theorem in this paper generalising these results, the method of proof is not a straightforward generalisation of the proof of stability for the braid group.

REMARK 1.2. Krannich [14] introduced a framework to study homological stability phenomena in the context of E_2 -algebras. This generalised a categorical framework of Randal-Williams and Wahl [21], which ‘automated’ parts of the homological stability proof for sequences of discrete groups. The sequence of monoids studied in this paper does not fit into

the categorical set up of [21], however, the classifying spaces of the sequence can be shown to assemble into an E_1 -module over an E_2 algebra, as in [14]. Combining the results of [14] with the high connectivity results established in this paper, our homological stability result with constant coefficients (Theorem A) can most likely be enhanced to one with abelian coefficients and coefficient systems of finite degree in the sense of [14, Section 4].

1.1. *Outline of proof*

The proof follows the outline of a standard homological stability argument, which we describe below for the benefit of the reader. We indicate where the new ingredients are used.

The proof of Theorem A requires the introduction of a semi-simplicial space \mathcal{A}_\bullet^n for each monoid in the sequence A_n^+ such that:

- (1) there exist homotopy equivalences $\mathcal{A}_p^n \simeq BA_{n-p-1}^+$ for every $p \geq 0$; and
- (2) there is a highly connected map from the geometric realisation of \mathcal{A}_\bullet^n to the classifying space BA_n^+ , which we denote $\|\phi_\bullet\|$

$$\|\mathcal{A}_\bullet^n\| \xrightarrow{\|\phi_\bullet\|} BA_n^+,$$

that is, $\|\phi_\bullet\|$ induces an isomorphism on a large range of homotopy groups.

The skeletal filtration of $\|\mathcal{A}_\bullet^n\|$ gives rise to a spectral sequence

$$E_{p,q}^1 = H_q(\mathcal{A}_p^n) \Rightarrow H_{p+q}(\|\mathcal{A}_\bullet^n\|).$$

From Point (1), it follows

$$E_{p,q}^1 = H_q(\mathcal{A}_p^n) = H_q(BA_{n-p-1}^+).$$

We prove that on the E^1 page under the above equality the differentials are given by either the zero map or the stabilisation map $s_*: H_q(BA_{n-p-1}^+) \rightarrow H_q(BA_{n-p}^+)$. Following this, applying the inductive hypotheses that previous monoids in the sequence satisfy stability gives that in a range (when q is small compared to n) the spectral sequence converges to $H_q(BA_{n-1}^+)$:

$$E_{p,q}^1 = H_q(BA_{n-p-1}^+) \Rightarrow H_{p+q}(\|\mathcal{A}_\bullet^n\|) = H_q(BA_{n-1}^+) \text{ in a range.}$$

The highly connected map of Point (2) above now gives that in a range the spectral sequence also converges to the homology of BA_n^+ , which completes the proof.

For sequences of discrete groups, a usual candidate for \mathcal{A}_\bullet^n would be built out of cosets of previous groups in the sequence. However, the fact that no tools exist for manipulating Artin cosets means that this approach cannot be taken. The coset theory developed in this paper for the corresponding sequence of Artin monoids is used to build \mathcal{A}_\bullet^n . The main obstacle in the proof is the high connectivity argument for Point 2 which follows a ‘union of chambers’ argument inspired by, but more involved than, work of Paris [19] and Davis [8].

1.2. *Organisation of the paper*

Sections 2 and 3 provide background on Coxeter groups and Artin groups, and the $K(\pi, 1)$ conjecture, respectively. Section 4 then introduces Artin monoids and develops a novel theory of ‘cosets’ and corresponding technical results. Following this, Section 5 details the required semi-simplicial background and particular monoid constructions used in the proof, some of which are new. Section 6 applies the theory of Section 4, and introduces notation used throughout the proof. Section 7 introduces the semi-simplicial space \mathcal{A}_\bullet^n and the map $\|\phi_\bullet\|$ described above. High connectivity of $\|\phi_\bullet\|$ is then the topic of Section 8, in which the general method of proof for the high connectivity argument is introduced, before the proof is split into several cases, due to the complexity of using Artin monoids. Finally, the spectral sequence argument and homological stability result are given in Section 9.

2. Coxeter groups and Artin groups

2.1. Coxeter groups

This section follows *The Geometry and Topology of Coxeter Groups* by Davis [8].

DEFINITION 2.1. A Coxeter matrix on a finite set of generators S is a symmetric matrix M indexed by elements of S , that is, with entries $m(s, t)$ in $\mathbb{N} \cup \infty$ for $\{s, t\}$ in $S \times S$. This matrix must satisfy:

- $m(s, s) = 1$ for all s in S ;
- $m(s, t) = m(t, s)$ must be either greater than 1, or ∞ , when $s \neq t$.

DEFINITION 2.2. A Coxeter matrix M with generating set S has an associated Coxeter group W , with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \rangle.$$

Here, $m(s, t) = \infty$ means there is no relation between s and t . We call (W, S) a Coxeter system. We adopt the convention that (W, \emptyset) is the trivial group.

REMARK 2.3. Note that the condition $m(s, s) = 1$ on the Coxeter matrix implies that the generators of the group are involutions, that is, $s^2 = e$ for all s in S .

DEFINITION 2.4. Define the length function on a Coxeter system (W, S)

$$\ell : W \rightarrow \mathbb{N}$$

to be the function which maps w in W to the minimum word length required to express w in terms of the generators.

DEFINITION 2.5. Define $\pi(a, b; k)$ to be the word of length k , given by the alternating product of a and b , that is,

$$\pi(a, b; k) = \overbrace{abab \dots}^{\text{length } k}$$

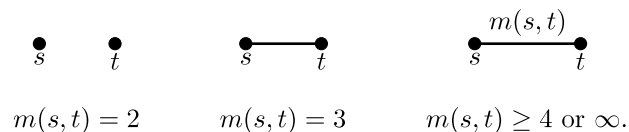
REMARK 2.6. The relations $(st)^{m(s,t)} = e$ can be rewritten as

$$\pi(s, t; m(s, t)) = \pi(t, s; m(s, t))$$

when $m(s, t) \neq \infty$. Therefore, the presentation of a Coxeter group W can also be given as

$$W = \left\langle S \mid \begin{array}{ll} (s)^2 = e & s \in S \\ \pi(s, t; m(s, t)) = \pi(t, s; m(s, t)) & s, t \in S \end{array} \right\rangle.$$

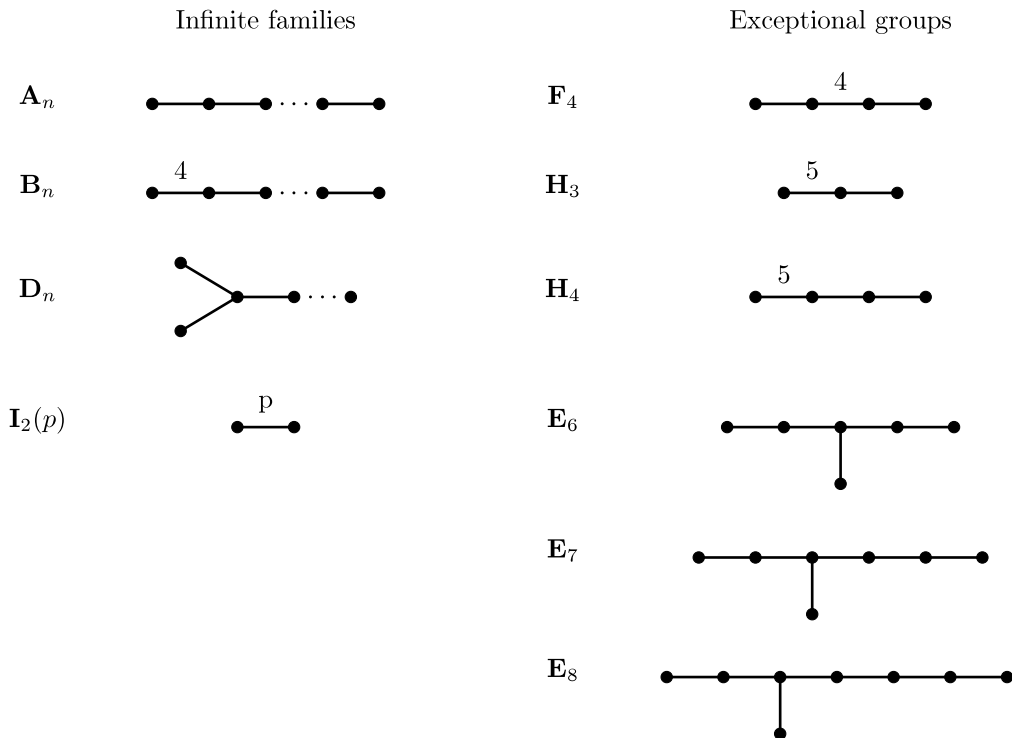
DEFINITION 2.7. Given a Coxeter matrix corresponding to a Coxeter system (W, S) , there is an associated graph called the Coxeter diagram, denoted \mathcal{D}_W . It is the graph with vertices indexed by the elements of the generating set S . Edges are drawn between the vertices corresponding to s and t in S when $m(s, t) \geq 3$ and labelled with $m(s, t)$ when $m(s, t) \geq 4$, as shown below:



When the diagram \mathcal{D}_W is connected, W is called an *irreducible* Coxeter group. The disjoint union of two diagrams gives the product of their corresponding Coxeter groups.

THEOREM 2.8 (Classification of finite Coxeter groups [7]). *A Coxeter group is finite if and only if it is a (direct) product of finitely many finite irreducible Coxeter groups.*

The following is a complete list of the diagrams corresponding to finite irreducible Coxeter groups.



DEFINITION 2.9. We say that a finite irreducible Coxeter group W is of *type \mathcal{D}* if its corresponding diagram is given by \mathcal{D} , and we denote this Coxeter group $W(\mathcal{D})$.

REMARK 2.10. The Coxeter group $W(A_n)$ is isomorphic to S_{n+1} , the symmetric group, which is the reflection group of the regular $(n + 1)$ -simplex.

DEFINITION 2.11. Let (W, S) be a Coxeter system. For each $T \subseteq S$, T generates a subgroup W_T such that (W_T, T) is a Coxeter system in its own right. We call subgroups that arise in this way *parabolic subgroups*. If the subgroup is finite, we call it a *spherical subgroup*.

2.2. Artin groups

This section follows Charney [6, Section 1] and notes by Paris [19].

Given a Coxeter system (W, S) , the corresponding Artin group is given by forgetting the involution relations, that is, setting $m(s, s) = \infty$.

DEFINITION 2.12. For every Coxeter system (W, S) , there is a corresponding *Artin system* (A_W, Σ) comprising an Artin group A_W with generating set

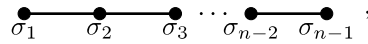
$$\Sigma := \{\sigma_s \text{ for } s \in S\}$$

and presentation

$$A_W = \langle \Sigma \mid s, t \in S, \pi(\sigma_s, \sigma_t; m(s, t)) = \pi(\sigma_t, \sigma_s; m(s, t)) \rangle.$$

We note that the Coxeter diagram \mathcal{D}_W contains all the information about the Artin group presentation.

EXAMPLE 2.13. The Artin group A_W corresponding to the Coxeter group $W \cong S_n$ is the braid group \mathcal{B}_n . The corresponding diagram \mathcal{D}_W is



where we relabel σ_{s_i} to σ_i for ease of notation. The presentation is therefore given by

$$\mathcal{B}_n = \left\langle \sigma_i \text{ for } s_i \in S \mid \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i - j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad 1 \leq i \leq (n - 2) \end{array} \right\rangle,$$

the standard presentation for the braid group on n strands.

EXAMPLE 2.14. When all possible edges in the Coxeter diagram \mathcal{D}_W are present and labelled with ∞ the corresponding Artin group is the free group on $|S|$ generators. The group has presentation

$$A_W = \langle \sigma_s \text{ for } s \in S \rangle.$$

EXAMPLE 2.15. When there are no edges in the Coxeter diagram \mathcal{D}_W , the corresponding Artin group is the free abelian group on $|S|$ generators. The group has presentation

$$A_W = \langle \sigma_s \text{ for } s \in S \mid \sigma_s \sigma_t = \sigma_t \sigma_s \ s \neq t \in S \rangle.$$

EXAMPLE 2.16. When all of the edges in the Coxeter diagram are labelled with ∞ , but not necessarily all possible edges are present (some $m(s, t)$ may be equal to 2), then the corresponding Artin group is called a *right-angled* Artin group, or RAAG.

DEFINITION 2.17. When the Coxeter group W is finite, that is, when its diagram \mathcal{D}_W is a disjoint union of diagrams from Proposition 2.8, then the corresponding Artin group A_W is called a *finite-type* Artin group, or a *spherical* Artin group.

Much of the known theory of Artin groups is concentrated around RAAGs and finite-type Artin groups, though we do not restrict ourselves to either of these families in our results. In general, little is known about Artin groups. For instance, the following properties hold for finite-type Artin groups [6].

- There exists a finite model for the classifying space $K(A_W, 1)$.
- A_W is torsion free.
- The centre of A_W is \mathbb{Z} , for A irreducible.
- A_W has solvable word and conjugacy problem.

To date these properties are not known for general Artin groups. In the next section, we consider the first point in detail.

3. The $K(\pi, 1)$ conjecture

This section introduces the $K(\pi, 1)$ conjecture, following [19].

In general, one can associate a hyperplane arrangement \mathcal{A} and associated complement $M(\mathcal{A})$ to each Coxeter group W , such that there is a free action of W on $M(\mathcal{A})$. When we consider this hyperplane complement modulo this W action, the corresponding quotient $M(\mathcal{A})/W$ has as its fundamental group the Artin group A_W . In some cases, this quotient space is known to be a $K(A_W, 1)$, in particular we recall Deligne's theorem for finite-type Artin groups.

THEOREM 3.1 (Deligne's theorem [9]). *For W a finite Coxeter group and A_W the associated Artin group, $M(\mathcal{A})/W$ is aspherical with fundamental group A_W , that is, $M(\mathcal{A})/W$ is a $K(A_W, 1)$.*

For arbitrary Artin groups, the $K(\pi, 1)$ conjecture was formulated by Arnol'd, Brieskorn, Pham and Thom, and states that an analogue of Deligne's theorem holds for all Artin groups. The analogue of the hyperplane complement was formulated by Vinberg. For a more detailed description, see Davis [8], notes by Paris [19] and the introduction to a paper on RAAGs by Charney [6].

REMARK 3.2. It is worth noting here a reformulation of the conjecture in terms of a finite-dimensional CW-complex called the *Salvetti complex*, denoted by $Sal(\mathcal{A})$ and introduced by Salvetti in [22], for a hyperplane arrangement \mathcal{A} in a finite-dimensional real vector space V . The Salvetti complex is defined in terms of cosets of finite subgroups of the Coxeter group [23]. Paris extended this definition to any infinite hyperplane arrangement in a non-empty convex cone I [19] and proved that $Sal(\mathcal{A})$ and $M(\mathcal{A})$ have the same homotopy type. The $K(\pi, 1)$ conjecture can therefore be restated as a conjecture about the Salvetti complex.

The $K(\pi, 1)$ conjecture has been proven for large classes of Artin groups [19]. However, the conjecture has not been proven for general Artin groups. We will apply a reformulation of the $K(\pi, 1)$ conjecture to our results, involving the Artin monoid A^+ and discussed in Section 4.

4. Artin monoids

The start of this section follows Jean Michel's *A note on words in braid monoids* [16] and Brieskorn and Saito's *Artin-Gruppen und Coxeter-Gruppen* [4]. Much of the material in Section 4.3 is new.

4.1. Definition and examples

DEFINITION 4.1. The *Artin monoid system* (A_W^+, Σ) associated to a Coxeter system (W, S) is given by the generating set Σ for the corresponding Artin system (A_W, Σ) , and the monoid with the same presentation as the Artin group A_W :

$$A_W^+ = \langle \Sigma \mid \pi(\sigma_s, \sigma_t; m(s, t)) = \pi(\sigma_t, \sigma_s; m(s, t)) \rangle^+.$$

Words in A_W^+ are therefore strings of letters for which the alphabet consists of σ_s in Σ .

REMARK 4.2. The group completion of A_W^+ is A_W .

EXAMPLE 4.3. The braid monoid \mathcal{B}_n^+ is the monoid associated to the Coxeter group S_n , the symmetric group, with group completion the braid group \mathcal{B}_n . Given the standard generating set for the symmetric group, the braid monoid consists of words in the braid group made from the positive generators σ_i . In terms of the braid diagrams these can be viewed as braids consisting of only positive twists.

DEFINITION 4.4. We call a submonoid M^+ of an Artin monoid A^+ a *parabolic submonoid* if the monoid M^+ is generated by the set $M^+ \cap \Sigma$. We call this generating set for the monoid Σ_M , giving a system (M^+, Σ_M) .

In this paper, by convention every submonoid of an Artin monoid considered will be a parabolic submonoid.

4.2. *Divisors in Artin monoids*

DEFINITION 4.5. Define the *length function* on an Artin monoid A^+ with system (A^+, Σ)

$$\ell : A^+ \rightarrow \mathbb{N}$$

to be the function which maps an element α in A^+ to the unique word length required to express α in terms of the generators in Σ .

REMARK 4.6. Note here that since there are no inverses in Artin monoids, multiplication corresponds to addition of lengths, that is, $\ell(ab) = \ell(a) + \ell(b)$ (ℓ is a monoid homomorphism).

DEFINITION 4.7. For elements α and β in an Artin monoid A^+ with system (A^+, Σ) , we say that $\alpha \preceq_R \beta$ if for some γ in A^+ we have $\beta = \gamma\alpha$, that is a word representing α appears on the right of some word representing β , in terms of the generating set Σ . We say that β is *right-divisible* by α , or alternatively that α *right-divides* β .

PROPOSITION 4.8 (Michel [16, Proposition 2.4]). *Artin monoids satisfy left and right cancellation, that is, for a, b and c in A^+ ,*

$$ab = ac \Rightarrow b = c$$

$$ba = ca \Rightarrow b = c.$$

We now consider work by Brieskorn and Saito in their 1972 paper *Artin-Gruppen und Coxeter-Gruppen* [4]. They consider notions of least common multiples and greatest common divisors of sets of elements in the Artin monoid. We are interested in the notion of least common multiple.

DEFINITION 4.9. Given a set of elements $\{g_j\}_{j \in J}$ in an Artin monoid A^+ with system (A^+, Σ) , a *common multiple* β is an element in A^+ which is right-divisible by all g_j . That is $g_j \preceq_R \beta$ for all j in J . A *least common multiple* of $\{g_j\}$ is a common multiple that right-divides all other common multiples. Let E be a set of elements in the Artin monoid A^+ . Denote the least common multiple (if it exists) of E by $\Delta(E)$. For α and β two elements in A^+ denote the least common multiple of α and β (if it exists) by $\Delta(\alpha, \beta)$.

REMARK 4.10. Should a least common multiple exist, it will be unique.

PROPOSITION 4.11 (Brieskorn and Saito [4, 4.1]). *A finite set of elements in an Artin monoid either has a least common multiple or no common multiple at all.*

REMARK 4.12. Since the relations in an Artin monoid have the same letters appearing on each side, the set of letters present in any word representing an element of an Artin monoid is fixed. Therefore, the notion of ‘letters appearing in an element’ is well defined.

LEMMA 4.13 (Brieskorn and Saito [4]). *For a fixed generating set, the letters arising in a least common multiple of a set of elements in an Artin monoid are only those letters which appear in the elements themselves.*

DEFINITION 4.14. Consider a submonoid M^+ of an Artin monoid A^+ , with system (M^+, Σ_M) . Given an element α in A^+ we define two end sets

$$\text{EndGen}_M(\alpha) = \{\sigma_s \mid s \in \Sigma_M, \sigma_s \preceq_R \alpha\}$$

$$\text{EndMon}_M(\alpha) = \{\beta \in M^+ \mid \beta \preceq_R \alpha\}.$$

REMARK 4.15. $\text{EndGen}_M(\alpha)$ is exactly the letters σ_s in Σ_M that a word representing α can end with, and $\text{EndMon}_M(\alpha)$ is exactly the elements in M^+ that a word representing α can end with. Note that $\text{EndGen}_M(\alpha)$ is a subset of $\text{EndMon}_M(\alpha)$, consisting of words of length 1 and $\text{EndMon}_M(\alpha) = \emptyset$ if and only if α has no right-divisors in M^+ .

4.3. Required theory

Much of the proof of Theorem A is concerned with algebraic manipulation of words in the Artin monoid. Here we introduce some technical definitions and lemmas used in the proof. We build up a theory of cosets in the case of Artin monoids, which is new unless cited.

LEMMA 4.16. *Given α in A^+ , and M^+ a submonoid of A^+ , the set $\text{EndMon}_M(\alpha)$ has a least common multiple $\Delta(\text{EndMon}_M(\alpha)) = \beta$ which lies in the submonoid M^+ . That is, there exists β in M^+ and γ in A^+ such that $\alpha = \gamma\beta$ for some words representing α, β, γ , and if β' in A^+ and γ' in A^+ satisfy $\alpha = \gamma'\beta'$, it follows that $\beta \preceq_R \beta'$.*

Proof. From Proposition 4.11 if a common multiple exists, then $\Delta(\text{EndMon}_M(\alpha))$ exists. We have that α itself is a common multiple of all elements in $\text{EndMon}_M(\alpha)$, by definition of $\text{EndMon}_M(\alpha)$. Furthermore, Lemma 4.13 notes that only letters appearing in $\text{EndMon}_M(\alpha)$ will appear in $\Delta(\text{EndMon}_M(\alpha))$. By definition, these are letters in M^+ and so $\Delta(\text{EndMon}_M(\alpha))$ lies in M^+ . \square

REMARK 4.17. For an element α in A^+ let $\Delta(\text{EndMon}_M(\alpha)) = \beta$. We write $\bar{\alpha}$ with respect to M^+ for the element $\bar{\alpha}$ in A^+ such that $\alpha = \bar{\alpha}\beta$. It will always be clear in the text with respect to which submonoid M^+ we are taking the reduction.

DEFINITION 4.18. For A^+ an Artin monoid and M^+ a submonoid, let $A^+(M)$ be the following set

$$A^+(M) = \{\bar{\alpha} \text{ with respect to } M^+ \mid \alpha \in A^+\}.$$

That is, $A^+(M)$ is the set of elements in A^+ which are not right-divisible by any element of M .

LEMMA 4.19. *For all α in A^+ and all β in M^+ , $\bar{\alpha} = \overline{\alpha\beta}$ where the reduction is taken with respect to M^+ .*

Proof. Let $\bar{\alpha} = \gamma$, so $\alpha = \gamma\eta$ for some η in M^+ , and $\text{EndMon}_M(\gamma) = \emptyset$, that is, γ has no right-divisors in M^+ . Then $\alpha\beta = \gamma\eta\beta$ and since η and β are both in M^+ , it follows that $\eta\beta \in \text{EndMon}_M(\alpha\beta)$. If $\eta\beta$ is the least common multiple of $\text{EndMon}_M(\alpha\beta)$, then $\overline{\alpha\beta} = \gamma = \bar{\alpha}$ so we are done. Suppose for a contradiction that $\eta\beta$ is not the least common multiple of $\text{EndMon}_M(\alpha\beta)$, and note that $\eta\beta$ is a right-divisor of the least common multiple. Then there exists some ζ in M^+ of length at least 1 such that $\zeta\eta\beta$

is the least common multiple of $\text{EndMon}_M(\alpha\beta)$. It follows that there exists a $\gamma' = \overline{\alpha\beta}$ with $\text{EndMon}_M(\gamma') = \emptyset$ and $\alpha\beta = \gamma'\zeta\eta\beta$. But $\alpha\beta = \gamma\eta\beta$ and it follows from cancellation that $\gamma = \gamma'\zeta$. Since ζ is in M^+ with length at least 1 it follows that $\zeta \in \text{EndMon}_M(\gamma)$ which contradicts $\text{EndMon}_M(\gamma) = \emptyset$. Therefore, $\eta\beta$ is the least common multiple of $\text{EndMon}_M(\alpha\beta)$ and it follows that $\overline{\alpha\beta} = \gamma = \overline{\alpha}$. \square

DEFINITION 4.20. Consider the relation \sim on A^+ given by

$$\alpha_1 \sim \alpha_2 \iff \alpha_1\beta_1 = \alpha_2\beta_2 \text{ for some } \beta_1 \text{ and } \beta_2 \text{ in } M^+,$$

where M^+ is a submonoid of A^+ . When we use this relation it will be made clear which submonoid M^+ is being considered. The relation \sim is symmetric and reflexive. Let \approx be the transitive closure of \sim . That is, $\alpha_1 \approx \alpha_2$ if there is a chain of elements in A^+ :

$$\alpha_1 \sim \tau_1 \sim \tau_2 \sim \dots \sim \tau_k \sim \alpha_2$$

for some k . Denote the equivalence class of α in A^+ under the relation \approx with respect to the submonoid M^+ as $[\alpha]_M$.

DEFINITION 4.21. Let $q : A^+(M) \rightarrow A^+ / \approx$ be the quotient map, taken with respect to the equivalence relation \approx .

LEMMA 4.22. The map q of Definition 4.21 is a bijection. That is for all α_1 and α_2 in A^+ :

$$[\alpha_1]_M = [\alpha_2]_M \iff \overline{\alpha_1} = \overline{\alpha_2}$$

Proof. (\Leftarrow) If $\overline{\alpha_1} = \overline{\alpha_2} = \gamma$ with respect to M^+ , then $\alpha_1 \sim \gamma \sim \alpha_2$ so it follows $\alpha_1 \approx \alpha_2$. (\Rightarrow) We want to show that if $\alpha_1 \approx \alpha_2$, then $\overline{\alpha_1} = \overline{\alpha_2}$. Since $\alpha_1 \approx \alpha_2$, there is a chain

$$\alpha_1 \sim \tau_1 \sim \tau_2 \sim \dots \sim \tau_k \sim \alpha_2$$

so if we show that $\overline{\eta} = \overline{\zeta}$ whenever $\eta \sim \zeta$ for η and ζ in A^+ it will follow that

$$\overline{\alpha_1} = \overline{\tau_1} = \overline{\tau_2} = \dots = \overline{\tau_k} = \overline{\alpha_2}.$$

Since $\eta \sim \zeta$ it follows that for some β_1 and β_2 in M^+ , $\eta\beta_1 = \zeta\beta_2$. Applying Lemma 4.19 gives

$$\overline{\eta} = \overline{\eta\beta_1} = \overline{\zeta\beta_2} = \overline{\zeta},$$

as required. \square

PROPOSITION 4.23. For M^+ a submonoid of A^+ , $A^+ \cong A^+(M) \times M^+$ as sets, via the bijection

$$p : A^+ \rightarrow A^+(M) \times M^+$$

$$\alpha \mapsto (\overline{\alpha}, \beta), \text{ where } \alpha = \overline{\alpha}\beta,$$

where $\beta = \Delta(\text{EndMon}_M(\alpha))$. This decomposition respects the right action of M^+ on A^+ , that is, M^+ acts trivially on the first factor and as right multiplication on the second.

Proof. To show p is surjective: consider $(\gamma, \beta) \in A^+(M) \times M^+$. Due to Lemma 4.19 for $\alpha \in A^+$ and any $\beta \in M^+$ we have $\overline{\alpha\beta} = \overline{\alpha}$. Therefore, $\gamma\beta$ satisfies $p(\gamma\beta) = (\gamma, \beta)$ since $\overline{\gamma\beta} = \overline{\gamma} = \gamma$ (we have $\gamma \in A^+(M)$ so $\text{EndMon}_p(\gamma) = \emptyset$). To show injectivity, suppose $p(\alpha_1) = p(\alpha_2)$, that is $(\overline{\alpha_1}, \beta_1) = (\overline{\alpha_2}, \beta_2)$. This translates to

$$\alpha_1 = \overline{\alpha_1}\beta_1 = \overline{\alpha_2}\beta_2 = \alpha_2,$$

therefore p is injective. Under this decomposition, the action of m in M^+ satisfies $p(\alpha \cdot m) = (\overline{\alpha}, \beta \cdot m)$, where $\alpha = \overline{\alpha}\beta$, again due to Lemma 4.19. \square

PROPOSITION 4.24 [16, 1.5]. *If generators s and t in S_M are in $\text{EndGen}_M(\alpha)$ for some α in A^+ , then $\Delta(s, t)$ lies in $\text{EndMon}_M(\alpha)$.*

LEMMA 4.25. *Consider a subset F of $\text{EndMon}_M(\alpha)$ for some submonoid M^+ of A^+ and some α in A^+ . Then $\Delta(F)$ is in $\text{EndMon}_M(\alpha)$.*

Proof. Since F is a subset of $\text{EndMon}_M(\alpha)$, which has a least common multiple, F has a common multiple so $\Delta(F)$ exists by Proposition 4.11. Since $\Delta(\text{EndMon}_M(\alpha))$ is a common multiple for $\text{EndMon}_M(\alpha)$, it is a common multiple for F . The element $\Delta(F)$ right-divides all other common multiples of F by definition. Therefore, $\Delta(F) \preceq_R \Delta(\text{EndMon}_M(\alpha))$ and it follows that $\Delta(F)$ is in $\text{EndMon}_M(\alpha)$. \square

DEFINITION 4.26. Recall from Remark 4.12 that the set of letters present in an element of an Artin monoid is well defined. We say elements α and β in an Artin monoid with system (A^+, Σ) *letterwise commute* if:

- the set of letters in Σ that α contains is disjoint from the set of letters that β contains, and
- each letter of Σ that α contains commutes with each letter of Σ that β contains.

LEMMA 4.27. *If β and γ are in $\text{EndMon}_M(\alpha)$ and β and γ letterwise commute, it follows that:*

- $\Delta(\beta, \gamma) = \beta\gamma = \gamma\beta$;
- $\Delta(\beta, \gamma) \in \text{EndMon}_M(\alpha)$.

Proof. Since β and γ letterwise commute, they contain distinct generators. From Remark 4.12 every word representative for β and γ contains the same set of letters. It follows each of these letters must appear in $\Delta(\beta, \gamma)$. If both β and γ have length 1, say $\beta = \sigma$ and $\gamma = \tau$ for generators σ and τ , then since the words letterwise commute it follows that σ commutes with τ . Therefore, since $\sigma\tau = \tau\sigma$ and both generators must appear in $\Delta(\beta, \gamma)$ it follows that

$$\Delta(\beta, \gamma) = \sigma\tau = \tau\sigma = \beta\gamma = \gamma\beta,$$

as required. Similarly, if $\beta = \sigma_1 \dots \sigma_k$ has length k , and $\gamma = \tau$ has length 1, then since the words contain distinct generators (which all must appear in the lowest common multiple) and $\tau\sigma_i = \sigma_i\tau \forall i$ it follows that

$$\Delta(\beta, \tau) = \Delta(\sigma_1 \dots \sigma_k, \tau) = (\sigma_1 \dots \sigma_k)\tau = \tau(\sigma_1 \dots \sigma_k) = \beta\tau = \tau\beta.$$

Suppose now that $\beta = \sigma_1 \dots \sigma_k$ has length k and $\gamma = \tau_1 \dots \tau_l$ has length l . It is true that $\beta \preceq_R \beta\gamma$ and $\gamma \preceq_R \beta\gamma$.

CLAIM. *If x in A^+ is a common multiple of β and γ , then $\beta\gamma = \gamma\beta$ is in $\text{EndMon}_M(x)$.*

Proof of claim. We proceed by induction on $\text{lg}(\gamma)$. The base case $\text{lg}(\gamma) = 1$ is covered above. As our inductive hypothesis, we suppose if $\text{lg}(\gamma') < l$ and γ' letterwise commutes with β , then $\Delta(\gamma', \beta) = \gamma'\beta = \beta\gamma'$, hence the claim holds. We prove the claim for

$$\beta = \sigma_1 \dots \sigma_k \text{ and } \gamma = \tau_1 \dots \tau_l.$$

Since x is a common multiple of β and γ , there exist y and z in A^+ such that

$$x = y\beta \text{ and } x = z\gamma = z\tau_1 \dots \tau_l.$$

In particular, β and $\gamma' = \tau_2 \dots \tau_l$ are in $\text{EndMon}_M(x)$, so

$$\Delta(\gamma', \beta) = \gamma' \beta = \beta \gamma' \in \text{EndMon}_M(x)$$

by the inductive hypothesis. Therefore, there exists $w \in A^+$ such that

$$x = z\gamma = z\tau_1 \dots \tau_l \text{ and } x = w\beta\gamma' = w\beta\tau_2 \dots \tau_l$$

which by cancellation of $\tau_2 \dots \tau_l$ gives $z\tau_1 = w\beta$. By the base case, since τ_1 and β are both in $\text{EndMon}_M(z\tau_1)$ so is $\Delta(\tau_1, \beta) = \tau_1\beta = \beta\tau_1$. Therefore, there exists $v \in A^+$ such that

$$z\tau_1 = v\beta\tau_1$$

and so by cancellation of τ_1 , it follows that $z = v\beta$. Reinserting this in the previous equation gives

$$x = z\gamma = v\beta\gamma$$

and so $\beta\gamma = \gamma\beta$ is in $\text{EndMon}_M(x)$ as required. □

LEMMA 4.28. *If words α , a and b in A^+ are such that $b \preceq_R \alpha a$ and a and b letterwise commute, then it follows that $b \preceq_R \alpha$.*

Proof. An equivalent way of writing $m \preceq_R n$ for m, n in A^+ is $m \in \text{EndMon}_A(n)$ where the end set is taken with respect to the full monoid A^+ . Since a and b are both in $\text{EndMon}_A(\alpha a)$ it follows that $\Delta(a, b)$ is in $\text{EndMon}_A(\alpha a)$, from Lemma 4.25. Since a and b letterwise commute, $\Delta(a, b) = ab = ba$ from Lemma 4.27. Therefore, ba is in $\text{EndMon}_A(\alpha a)$ and, by cancellation of a , b is in $\text{EndMon}_A(\alpha)$ as required. □

4.4. Relation to the $K(\pi, 1)$ conjecture

In 2002, Dobrinskaya published a paper relating the classifying space of the Artin monoid BA_W^+ to the $K(\pi, 1)$ conjecture. This was later translated into English as *Configuration Spaces of Labelled Particles and Finite Eilenberg - MacLane Complexes* [10]. The main result of the paper was the following.

THEOREM 4.29 (Dobrinskaya [10, Theorem 6.3]). *Given an Artin group A_W and its associated monoid A_W^+ , the $K(\pi, 1)$ conjecture holds if and only if the natural map between their classifying spaces, $BA_W^+ \rightarrow BA_W$ is a homotopy equivalence.*

This theorem has been reproved using a different Morse-theoretic approach by Ozornova [17] and her result has in turn been strengthened by Paolini [18].

5. Semi-simplicial constructions with monoids

This section is split into three subsections. The first introduces background semi-simplicial theory before the second introduces theory for generic monoids and submonoids, including some new results. The third subsection focuses on Artin monoids and contains results required later in the proof.

5.1. Semi-simplicial objects

This subsection consists of the required background and follows Ebert and Randal-Williams [12].

DEFINITION 5.1 [12, 1.1]. Let Δ denote the category which has as objects the non-empty finite ordered sets $[n] = \{0, 1, \dots, n\}$, and as morphisms monotone increasing functions. These functions are generated by the basic functions which act on the ordered sets as follows:

$$\begin{aligned} D^i &: [n] \rightarrow [n + 1] \text{ for } 0 \leq i \leq n \\ &\{0, 1, \dots, n\} \mapsto \{0, 1, \dots, \widehat{i}, \dots, n + 1\} \\ S^i &: [n + 1] \rightarrow [n] \text{ for } 0 \leq i \leq n \\ &\{0, 1, \dots, n + 1\} \mapsto \{0, 1, \dots, i, i, \dots, n\}. \end{aligned}$$

The opposite category Δ^{op} is known as the simplicial category. We denote the opposite of the maps D^i by ∂_i and the opposite of the maps S^i by s_i . We call these the *face maps* and the *degeneracy maps*, respectively.

Let $\Delta_{inj} \subset \Delta$ be the subcategory of Δ which has the same objects but only the injective monotone maps as morphisms, generated by the D_i . The opposite category Δ_{inj}^{op} is known as the semi-simplicial category and its morphisms are therefore generated by the face maps ∂_i .

DEFINITION 5.2 [12, 1.1]. A *simplicial object* in a category \mathcal{C} is a covariant functor $X_\bullet : \Delta^{op} \rightarrow \mathcal{C}$. A *semi-simplicial object* is a functor $X_\bullet : \Delta_{inj}^{op} \rightarrow \mathcal{C}$. We denote $X_\bullet([n])$ by X_n . A *(semi-)simplicial map* $f : X_\bullet \rightarrow Y_\bullet$ is a natural transformation of functors, and in particular has components $f_n : X_n \rightarrow Y_n$. Simplicial objects in \mathcal{C} form a category denoted \mathbf{sC} , and semi-simplicial objects a category denoted \mathbf{ssC} . When \mathcal{C} equals *Set* a (semi-)simplicial object is called a *(semi-)simplicial set* and when \mathcal{C} equals *Top* it is called a *(semi-)simplicial space*.

REMARK 5.3. A semi-simplicial object in a category \mathcal{C} is equivalent to the following data.

- (a) An object X_p in \mathcal{C} , for $p \geq 0$.
- (b) Morphisms in \mathcal{C} $\partial_i^p : X_p \rightarrow X_{p-1}$ for $0 \leq i \leq p$ and all $p \geq 0$ called *face maps*, which satisfy the following *simplicial identities*

$$\partial_i^{p-1} \partial_j^p = \partial_{j-1}^{p-1} \partial_i^p \text{ if } i < j.$$

DEFINITION 5.4 [12, 1.3]. An augmented semi-simplicial object in \mathcal{C} is a triple $(X_\bullet, X_{-1}, \epsilon_\bullet)$ such that X_\bullet is a semi-simplicial object in \mathcal{C} , X_{-1} is an object of \mathcal{C} and ϵ_\bullet is a family of morphisms such that $\epsilon_p : X_p \rightarrow X_{-1}$ and $\epsilon_{p-1} \circ \partial_i = \epsilon_p$ for all $p \geq 1$ and $0 \leq i \leq p$.

EXAMPLE 5.5 [12, 1.2]. The standard n -simplex has two equivalent manifestations: as a simplicial object in *Set* and as an object in *Top*. When viewed as a simplicial set the standard n -simplex is denoted Δ_\bullet^n and is defined via the functor $\Delta_m^n = \Delta_\bullet^n([m]) = \text{hom}_\Delta([m], [n])$ for all $[m]$ in Δ^{op} . When viewed as an object in *Top* the standard n -simplex is denoted $|\Delta^n|$ and defined to be

$$|\Delta^n| := \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1 \text{ and } t_i \geq 0 \forall i \right\}.$$

One can associate to a morphism $\phi : [n] \rightarrow [m]$ in Δ a continuous map

$$\begin{aligned} \phi_* &: |\Delta^n| \rightarrow |\Delta^m| \\ (t_0, \dots, t_n) &\mapsto (s_0, \dots, s_m), \text{ where } s_j = \sum_{\phi(i)=j} t_i. \end{aligned}$$

That is, morphisms send the j th vertex of the simplex $|\Delta^n|$ to the $\phi(j)$ th vertex of $|\Delta^m|$ and extend linearly. Under this viewpoint the map D_*^i sends $|\Delta^n|$ to the i th face of $|\Delta^{n+1}|$ and the map S_*^i collapses together the i th and $(i + 1)$ st vertices of $|\Delta^{n+1}|$ to give a map to $|\Delta^n|$.

A tuple $(\partial_{i_1}^{p-1}, \partial_{i_2}^{p-2}, \dots, \partial_{i_k}^{p-k})$ denotes the application of several face maps in a row, where $\partial_{i_1}^{p-1}$ is the first face map to be applied, followed by $\partial_{i_2}^{p-2}$, etc. For ease of notation we dispense with superscripts, writing the tuple as $(\partial_{i_1}, \partial_{i_2}, \dots, \partial_{i_k})$ and assuming the domain and targets are such that the composite map is defined.

LEMMA 5.6. *With the above notation, the tuple of face maps can be written such that $i_{j+1} \geq i_j$ for all j .*

Proof. Suppose $i_{j+1} < i_j$ in the tuple $(\partial_{i_1}, \partial_{i_2}, \dots, \partial_{i_k})$. The simplicial identities show $\partial_{i_{j+1}} \partial_{i_j} = \partial_{i_j-1} \partial_{i_{j+1}}$ since $i_{j+1} < i_j$. Therefore,

$$(\partial_{i_1}, \partial_{i_2} \dots, \partial_{i_j}, \partial_{i_{j+1}}, \dots, \partial_{i_k}) = (\partial_{i_1}, \partial_{i_2} \dots, \partial_{i_{j+1}}, \partial_{i_j-1}, \dots, \partial_{i_k}).$$

Since $i_{j+1} < i_j$, it follows that $i_j - 1 \geq i_{j+1}$. Relabelling $i_j := i_{j+1}$ and $i_{j+1} := i_j - 1$ gives $(\partial_{i_1}, \partial_{i_2} \dots, \partial_{i_j}, \partial_{i_{j+1}}, \dots, \partial_{i_k})$ such that $i_{j+1} \geq i_j$. This procedure reduces the sum $\sum_{j=1}^k i_j$ by one, and therefore upon iteration must terminate. Applying this process enough times gives $i_{j+1} \geq i_j$ for all j . \square

DEFINITION 5.7 [12, 1.2]. The *geometric realisation* of a semi-simplicial set or space is the topological space denoted by $\|X_\bullet\|$ and defined to be

$$\|X_\bullet\| := \coprod_{n \geq 0} X_n \times |\Delta^n| / \sim,$$

where \sim is generated by $(x, t) \sim (y, u)$, whenever $\partial_i(x) = y$ and $D^i(u) = t$.

The geometric realisation is an example of a coequaliser or colimit (see Dugger [11]).

DEFINITION 5.8. Given a semi-simplicial map $f_\bullet : X_\bullet \rightarrow Y_\bullet$ there is an induced map $\|f_\bullet\| : \|X_\bullet\| \rightarrow \|Y_\bullet\|$ which we call the *geometric realisation of the semi-simplicial map f_\bullet* .

DEFINITION 5.9 [12, 1.4]. A *bi-semi-simplicial* object in a category \mathcal{C} is a functor $X_{\bullet\bullet} : (\Delta_{inj} \times \Delta_{inj})^{op} \rightarrow \mathcal{C}$. We write $X_{p,q} = X_{\bullet\bullet}([p], [q])$. We write the image of the standard face maps in each simplicial direction $(\partial_i \times \text{id})$ and $(\text{id} \times \partial_j)$, as $\partial_{i,\bullet}$ and $\partial_{\bullet,j}$. We note that

$$(\partial_i \times \partial_j) = (\partial_{i,\bullet} \circ \partial_{\bullet,j}) = (\partial_{\bullet,j} \circ \partial_{i,\bullet}) : X_{p,q} \rightarrow X_{(p-1),(q-1)}$$

and we denote this map $\partial_{i,j}$. When \mathcal{C} is equal to *Top* the bi-semi-simplicial object is called a *bi-semi-simplicial space*.

REMARK 5.10. A bi-semi-simplicial space can be viewed as a semi-simplicial object in *ssTop* in two ways:

- (1) $X_{\bullet,q} : [p] \mapsto (X_\bullet : [q] \mapsto X_{p,q})$ with face maps $\partial_{i,\bullet}$;
- (2) $X_{p,\bullet} : [q] \mapsto (X_\bullet : [p] \mapsto X_{p,q})$ with face maps $\partial_{\bullet,j}$.

DEFINITION 5.11 [12, 1.2]. Given a bi-semi-simplicial space $X_{\bullet\bullet}$ we define its *geometric realisation* to be

$$\|X_{\bullet\bullet}\| = \coprod_{p,q \geq 0} X_{p,q} \times |\Delta^p| \times |\Delta^q| / \sim,$$

where \sim is generated by $(x, t_1, t_2) \sim (y, u_1, u_2)$ whenever $(\partial_{i,j})(x) = y$, $D^i(u_1) = t_1$ and $D^j(u_2) = t_2$. This is equivalent to taking the geometric realisation of the semi-simplicial space first in the p direction, followed by the q direction, or in the q direction followed by the p direction. This is due to the following homeomorphisms [12, 1.9 and 1.10] :

$$\|X_{\bullet, \bullet}\| \cong \|X_{\bullet, q} : [p] \mapsto \|X_{\bullet} : [q] \mapsto X_{p, q}\| \cong \|X_{p, \bullet} : [q] \mapsto \|X_{\bullet} : [p] \mapsto X_{p, q}\|.$$

5.2. Semi-simplicial constructions using monoids and submonoids

The following description of the geometric bar construction and related definitions loosely follows Chapter 7 of May's *Classifying spaces and fibrations* [15]. In this section, we view monoids and groups as discrete spaces.

DEFINITION 5.12. Let M be a monoid and let X and Y be spaces with a left and right action of M , respectively. Then the *bar construction* denoted $B(Y, M, X)$ is the geometric realisation of the semi-simplicial space $B_{\bullet}(Y, M, X)$ given by

$$B_n(Y, M, X) = Y \times M^n \times X.$$

Elements in $B_n(Y, M, X)$ are written as $y[g_1, \dots, g_n]x$ for $y \in Y$, $g_i \in M$ for $1 \leq i \leq n$ and $x \in X$. Face maps are given by

$$\partial_i(y[g_1, \dots, g_n]x) = \begin{cases} yg_1[g_2, \dots, g_n]x & \text{if } i = 0 \\ y[g_1, \dots, g_i g_{i+1}, \dots, g_n]x & \text{if } 1 \leq i \leq n-1 \\ y[g_1, \dots, g_{n-1}]g_n x & \text{if } i = n. \end{cases}$$

DEFINITION 5.13. Consider the bar construction $B(*, M, Y)$ for Y a space with an action of the monoid M on the left and $*$ a point on which M acts trivially. Define this to be the *homotopy quotient* of Y over M (or M under Y) and denote it $B(*, M, Y) =: M \parallel Y$. This is the geometric realisation of the semi-simplicial space $B_{\bullet}(*, M, Y)$ given by

$$B_j(*, M, Y) = * \times M^j \times Y.$$

Elements are written as $[m_1, \dots, m_j]y$ for m_i in M for $1 \leq i \leq j$ and y in Y . Face maps are given by

$$\partial_i([m_1, \dots, m_j]y) = \begin{cases} [m_2, \dots, m_j]y & \text{if } i = 0 \\ [m_1, \dots, m_i m_{i+1}, \dots, m_j]y & \text{if } 1 \leq i \leq j-1 \\ [m_1, \dots, m_{j-1}]m_j y & \text{if } i = j. \end{cases}$$

In the situation of a monoid M acting on a space Y on the right we define the homotopy quotient to be $B(Y, M, *) =: Y \parallel M$.

EXAMPLE 5.14. Consider the bar construction $B(*, N, M)$, for N a submonoid of M acting on M on the left, by left multiplication, and $*$ a point, on which N necessarily acts trivially. Then the homotopy quotient of M over N is

$$B(*, N, M) = N \parallel M.$$

This is the geometric realisation of the semi-simplicial space $B_{\bullet}(*, N, M)$ given by

$$B_j(*, N, M) = * \times N^j \times M.$$

Elements are written as $[n_1, \dots, n_j]m$ for n_i in N for $1 \leq i \leq j$ and m in M . Face maps are given by

$$\partial_i([n_1, \dots, n_j]m) = \begin{cases} [n_2, \dots, n_j]m & \text{if } i = 0 \\ [n_1, \dots, n_i n_{i+1}, \dots, n_j]m & \text{if } 1 \leq i \leq j - 1 \\ [n_1, \dots, n_{j-1}]n_j m & \text{if } i = j. \end{cases}$$

We can build a similar homotopy quotient for a submonoid N acting on M on the right by right multiplication. Then the associated homotopy quotient is the geometric realisation $B(M, N, *) = M // N$.

LEMMA 5.15. *The homotopy quotient of a group G or monoid M under a point $*$ is a model for the classifying space of the group or monoid, that is, $BG \simeq G // * \simeq * // G$ and $BM \simeq M // * \simeq * // M$.*

Proof. Writing down the simplices and face maps for the homotopy quotients $G // *$ and $G // *$ gives exactly the simplices and face maps for the *standard resolution* or *bar resolution* of G , which is a model for BG (see, for example, [5]). This holds similarly for the monoid M (see, for example, [15, p. 31]). \square

LEMMA 5.16. *For a monoid M , $M // M \simeq *$.*

Proof. This is a consequence of [12, Lemma 1.12] using the augmentation to a point. \square

LEMMA 5.17. *Let N be a monoid and S be a space with right N action. Suppose S can be decomposed as $S \cong X \times Y$ and, under this decomposition, the action of N restricts to a right action on the Y component and trivial action on the X component. Then the map given by the geometric realisation of the levelwise map on the bar construction*

$$\begin{aligned} B_p((X \times Y), N, *) &\rightarrow X \times B_p(Y, N, *) \\ (x, y)[n_1, \dots, n_p] &\mapsto (x, y[n_1, \dots, n_p]) \end{aligned}$$

for $x \in X, y \in Y$ and $n_i \in N$ for all i is a homotopy equivalence. That is, the homotopy quotient satisfies

$$S // N \cong (X \times Y) // N \simeq X \times (Y // N),$$

where the homotopy equivalence is given by the geometric realisation of the levelwise map on the bar construction

$$\begin{aligned} B_p((X \times Y), N, *) &\rightarrow X \times B_p(Y, N, *) \\ (x, y)[n_1, \dots, n_p] &\mapsto (x, y[n_1, \dots, n_p]) \end{aligned}$$

for $x \in X, y \in Y$ and $n_i \in N$ for all i .

Proof. The homotopy quotient $S // N$ is the geometric realisation of the simplicial space $B_\bullet(S, N, *)$ with j -simplices given by

$$B_j(S, N, *) = S \times N^j$$

and face maps given by Definition 5.13, the first face map ∂_1 encoding the right action of N on S . Under the decomposition $S \cong X \times Y$ the j -simplices are given by

$$B_j(S, N, *) \cong (X \times Y) \times N^j \cong X \times (Y \times N^j),$$

where the second isomorphism highlights that the action of N on S can be restricted to a right action on Y , since the action is trivial on the X component. Note that the second factor is precisely the j -simplices in $B_j(Y, N, *)$, and since the face maps act trivially on the X factor, the face maps in $B_j(S, N, *)$ induce face maps in $B_j(Y, N, *)$ under the decomposition. The proof is concluded by taking the geometric realisation of $B_\bullet(S, N, *)$ and the geometric realisation of $X \times B_\bullet(Y, N, *)$, noting that $\|X \times B_\bullet(Y, N, *)\| \simeq X \times \|B_\bullet(Y, N, *)\|$. \square

5.3. *Semi-simplicial constructions for Artin monoids*

Given an Artin monoid A^+ and a parabolic submonoid M^+ , recall from Section 4 that $A^+(M)$ is the set of elements in A^+ which do not end in elements in M^+ and there is a decomposition as sets (Proposition 4.23), $A^+ \cong A^+(M) \times M^+$. This decomposition maps α in A^+ to $(\bar{\alpha}, \beta)$, where $\alpha = \bar{\alpha}\beta$ (as defined in Remark 4.17) and the right action of M^+ on A^+ descends to a trivial action on $A^+(M)$ and a right action on M^+ .

In this section, we view monoids and sets as discrete spaces.

PROPOSITION 5.18. *The map*

$$A^+ \parallel M^+ \rightarrow A^+(M)$$

*defined levelwise on the bar construction $B_\bullet(A^+, M^+, *)$ by*

$$\begin{aligned} B_p(A^+, M^+, *) &\rightarrow A^+(M) \\ \alpha[m_1, \dots, m_p] &\mapsto \bar{\alpha} \end{aligned}$$

is a homotopy equivalence.

Proof. From Proposition 4.23, $A^+ \cong A^+(M) \times M^+$ as sets, hence as discrete spaces, and this decomposition respects the right action of M^+ on A^+ . Then

$$\begin{aligned} A^+ \parallel M^+ &= (A^+(M) \times M^+) \parallel M^+ \\ &\simeq A^+(M) \times (M^+ \parallel M^+) \\ &\simeq A^+(M) \times * \\ &= A^+(M), \end{aligned}$$

where the first homotopy equivalence uses Lemma 5.17 and the second homotopy equivalence uses Lemma 5.16. The levelwise map given by the composition of the maps in these two lemmas is precisely the map in the statement. \square

PROPOSITION 5.19. *Let A^+ be a monoid and M^+ be a submonoid. Consider two maps f and $g : A^+ \parallel A^+ \rightarrow A^+ \parallel A^+$ which are both equivariant with respect to the action of M^+ on the right of $A^+ \parallel A^+$. Then there exists an M^+ equivariant homotopy between the two maps.*

Proof. Denote the set of k -cells in $A^+ \parallel A^+$ as $(A^+ \parallel A^+)_k$. Let the k -cell of $A^+ \parallel A^+$ corresponding to geometric realisation of the k -simplex $[p_1, \dots, p_k]a$ of $B_k(*, A^+, A^+)$ (as in Definition 5.13) be denoted by the tuple (p_1, \dots, p_k, a) , with p_i and a in A^+ . There is a right action of A^+ on the k -cells given by

$$(p_1, \dots, p_k, a) \cdot \mu = (p_1, \dots, p_k, a\mu).$$

Define the set of *elementary k -cells* to be those with tuple (p_1, \dots, p_k, e) , where e is the identity element in the monoid, and denote this cell $D(p_1, \dots, p_k)$. Then

every k -cell is uniquely determined by an elementary k -cell and an element a in A^+ , since $(p_1, \dots, p_k, a) = D(p_1, \dots, p_k) \cdot a$. The isomorphism of Proposition 4.23 shows that $A^+ = A^+(M) \times M^+$ and we let $a = \bar{a}m$ under this decomposition. Then we get the following description for k -cells:

$$(A^+ \parallel A^+)_k \cong \bigcup_{(p_1, \dots, p_k)} D(p_1, \dots, p_k) \times A^+ \cong \bigcup_{(p_1, \dots, p_k)} D(p_1, \dots, p_k) \times (A^+(M) \times M^+)$$

$$(p_1, \dots, p_k, a) \longmapsto (D(p_1, \dots, p_k), a) \longmapsto (D(p_1, \dots, p_k), (\bar{a}, m)).$$

Let f_k be the restriction of the map f to the k -cells of $A^+ \parallel A^+$ and similarly for g_k . We first define an equivariant homotopy between f_0 and g_0 . Under the above decomposition, $(A^+ \parallel A^+)_0 \cong (A^+(M) \times M^+)$. Consider $f_0(\bar{\alpha})$ and $g_0(\bar{\alpha})$ in $A^+ \parallel A^+$ for $\bar{\alpha}$ in $A^+(M)$. Then since $A^+ \parallel A^+ \simeq *$ by Lemma 5.16 it follows that there exists a path between $f_0(\bar{\alpha})$ and $g_0(\bar{\alpha})$: call this $h_0(\bar{\alpha}, t)$ for $t \in [0, 1]$. Extend this homotopy to all 0-cells by setting $h_0(\bar{\alpha}m, t) = h_0(\bar{\alpha}, t) \cdot m$ for all m in M^+ . Then, since f_0 and g_0 are M^+ equivariant,

$$h_0(\bar{\alpha}m, 0) = h_0(\bar{\alpha}, 0) \cdot m = f_0(\bar{\alpha}) \cdot m = f_0(\bar{\alpha}m)$$

and similarly

$$h_0(\bar{\alpha}m, 1) = h_0(\bar{\alpha}, 1) \cdot m = g_0(\bar{\alpha}) \cdot m = g_0(\bar{\alpha}m).$$

The homotopy $h_0(x, t)$ is M^+ equivariant, since $h_0(x, t) \cdot \mu = h_0(x\mu, t)$ for μ in M^+ .

Now assume that we have built an equivariant homotopy $h_{k-1}(x, t)$ on the $(k-1)$ -skeleton and we show how to extend it to the k -cells. The homotopy $h_{k-1}(x, t)$ satisfies $h_{k-1}(x, 0) = f_{k-1}(x)$ and $h_{k-1}(x, 1) = g_{k-1}(x)$. For some $\bar{\alpha}$ in $A^+(M)$, consider the k -cell $D(p_1, \dots, p_k) \cdot \bar{\alpha}$. Then its boundary consists of $(k-1)$ -cells and it follows that h_{k-1} defines a homotopy

$$(\partial(D(p_1, \dots, p_k)) \cdot \bar{\alpha}) \times I \rightarrow A^+ \parallel A^+$$

and the maps f_k and g_k also define maps

$$f_k : ((D(p_1, \dots, p_k)) \cdot \bar{\alpha}) \times \{0\} \rightarrow A^+ \parallel A^+$$

$$g_k : ((D(p_1, \dots, p_k)) \cdot \bar{\alpha}) \times \{1\} \rightarrow A^+ \parallel A^+.$$

The union of these three maps defines a map from $\partial((D(p_1, \dots, p_k) \cdot \bar{\alpha}) \times I)$ to $A^+ \parallel A^+$, but this boundary is a $(k-1)$ -sphere and so, since $A^+ \parallel A^+$ is contractible the $(k-1)$ -sphere bounds a (k) -disk. We can compatibly extend the map over this disk to create the required homotopy

$$h_k : (D(p_1, \dots, p_k) \cdot \bar{\alpha}) \times I \rightarrow A^+ \parallel A^+$$

which agrees on the boundary with the three maps above. Now define h_k on any k -cell $D(p_1, \dots, p_k) \cdot \bar{\alpha}m$ by the following: for x in $D(p_1, \dots, p_k) \cdot \bar{\alpha}$ we set

$$h_k(x \cdot m, t) = h_k(x, t) \cdot m.$$

Then by construction h_k is M^+ equivariant and, since both f_k and g_k are M^+ equivariant, h_k satisfies $h_k(x, 0) = f_k$ and $h_k(x, 1) = g_k$. □

DEFINITION 5.20. Given a monoid M and two submonoids N_1 and N_2 we can define the *double homotopy quotient* $N_1 \parallel M \parallel N_2$ to be the geometric realisation of the bi-semi-simplicial space (recall Definition 5.9) defined by taking the two simplicial directions arising from the bar constructions $B_\bullet(*, N_1, M)$ and $B_\bullet(M, N_2, *)$. The p, q level of the associated bi-semi-simplicial space $X_{\bullet, \bullet}$ has simplices

$$X_{p,q} = N_1^p \times M \times N_2^q$$

and face maps inherited from $B_\bullet(*, N_1, M)$ in the p direction $(\partial_{p, \bullet})$ and $B_\bullet(M, N_2, *)$ in the q direction $(\partial_{\bullet, q})$. Then $[n_1, \dots, n_p]m[n'_1, \dots, n'_q]$ represents an element in the p, q level, where n_i

in N_1 and n'_j in N_2 for $1 \leq i \leq p$ and $1 \leq j \leq q$. We note that the face maps on the left and right commute, since the only maps which act on the same coordinates are $\partial_{p,\bullet}$ in the p direction and $\partial_{\bullet,0}$ in the q direction and these commute:

$$\begin{aligned} \partial_{p,\bullet}(\partial_{\bullet,0}([n_1, \dots, n_p]m[n'_1, \dots, n'_q])) &= \partial_{p,\bullet}([n_1, \dots, n_p]mn'_1[n'_2, \dots, n'_q]) \\ &= [n_1, \dots, n_{p-1}]n_p mn'_1[n'_2, \dots, n'_q] \\ &= \partial_{\bullet,0}(\partial_{p,\bullet}([n_1, \dots, n_p]m[n'_1, \dots, n'_q])). \end{aligned}$$

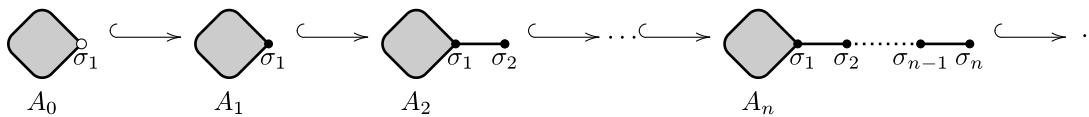
6. Preliminaries concerning the sequence of Artin monoids

This section introduces notation used throughout the remainder of the proof.

We consider the sequence of Artin monoids and inclusions

$$A_0^+ \hookrightarrow A_1^+ \hookrightarrow A_2^+ \hookrightarrow \dots \hookrightarrow A_n^+ \hookrightarrow \dots \tag{1}$$

with Artin monoid systems (A_n^+, Σ_n) given by the following diagrams. Here, the Artin Monoid A_i corresponds to the Coxeter group W_i , as defined in [13], and so we denote the corresponding Coxeter diagrams \mathcal{D}_{W_i} .



DEFINITION 6.1. Let (A_0, Σ_0) be the Artin system corresponding to the Coxeter diagram \mathcal{D}_{W_1} , but with the vertex σ_1 and all edges which have vertex σ_1 at one end removed. We depict the diagram as above. Note that $A_0 \hookrightarrow A_1$.

REMARK 6.2. With the generating sets corresponding to the above sequence of diagrams, for all p every generator and hence every word in the monoid A_p^+ commutes with σ_j for $j \geq p + 2$.

We now apply the theory developed in Section 4.3 to the specific case of a monoid A_n^+ in the sequence of monoids and inclusions (1) and the submonoid of A_n^+ , given by a previous monoid in the sequence A_p^+ , where $p < n$. We adopt the following notation for the remainder of this paper. The generating set of A_n^+ will always be given by Σ_n , the generating set specified by the diagram \mathcal{D}_{W_n} .

- Let $\text{EndMon}_p(\alpha) = \text{EndMon}_{A_p}(\alpha)$ and $\text{EndGen}_p(\alpha) = \text{EndGen}_{A_p}(\alpha)$ for α in A_n^+ , as in Definition 4.14. Then

$$\text{EndGen}_p(\alpha) = \{\sigma_s \mid s \in S_{A_p^+}, \sigma_s \preceq_R \alpha\}$$

$$\text{EndMon}_p(\alpha) = \{\beta \in A_p^+ \mid \beta \preceq_R \alpha\}.$$

- Let $A^+(n; p)$ be the set $A^+(M)$ for $A^+ = A_n^+$ and $M = A_p^+$ as in Definition 4.18 (this is the set of elements in A_n^+ that do not end in a non-trivial element in A_p^+).

- Let the equivalence class of α in A_n^+ under the relation \approx with respect to the submonoid A_p^+ (Definition 4.20) be denoted $[\alpha]_p$ as opposed to $[\alpha]_{A_p}$. Then $[\alpha]_p$ is the equivalence class of α under \approx , the transitive closure of the relation \sim on A_n^+ given by

$$\alpha_1 \sim \alpha_2 \iff \alpha_1 \beta_1 = \alpha_2 \beta_2 \text{ for some } \beta_1 \text{ and } \beta_2 \text{ in } A_p^+.$$

Then we have from Lemma 4.22 that the equivalence classes under \approx with respect to the submonoid A_p^+ are in one to one correspondence with the set $A^+(n; p)$. Recall from Remark 4.17

that if β is the least common multiple of $\text{EndMon}_p(\alpha)$, then we define $\bar{\alpha}$ in A_n^+ to be the element such that $\alpha = \bar{\alpha}\beta$. Then $A^+(n; p)$ is the set of all such $\bar{\alpha}$ and for all α_1 and α_2 in A_n^+ :

$$[\alpha_1]_p = [\alpha_2]_p \iff \bar{\alpha}_1 = \bar{\alpha}_2.$$

We also have from Proposition 4.23 the decomposition

$$A_n^+ \cong A^+(n; p) \times A_p^+ \text{ for all } p < n.$$

7. The semi-simplicial space \mathcal{A}_\bullet^n

We now build the semi-simplicial space \mathcal{A}_\bullet^n as promised in Section 1.1.

DEFINITION 7.1. Define a semi-simplicial space \mathcal{C}_\bullet^n by, for $0 \leq p \leq (n - 1)$, setting levels \mathcal{C}_p^n to be the discrete space of equivalence classes A_n^+ / \approx where the equivalence relation is taken with respect to the submonoid A_{n-p-1}^+ , that is, \approx is the transitive closure of the relation \sim on A_n^+ given by

$$\alpha_1 \sim \alpha_2 \iff \alpha_1\beta_1 = \alpha_2\beta_2 \text{ for some } \beta_1 \text{ and } \beta_2 \text{ in } A_{n-p-1}^+.$$

Face maps are given by

$$\begin{aligned} \partial_k^p : \mathcal{C}_p^n &\rightarrow \mathcal{C}_{p-1}^n \text{ for } 0 \leq k \leq p \\ \partial_k^p : [\alpha]_{n-p-1} &\mapsto [\alpha(\sigma_{n-p+k}\sigma_{n-p+k-1} \cdots \sigma_{n-p+1})]_{n-p}. \end{aligned}$$

For example, ∂_0^p acts on the equivalence class representative by right multiplication by e , and ∂_p^p acts by right multiplication by $\sigma_n \cdots \sigma_{n-p-1}$. The motivation for this choice of face maps follows Hepworth, as discussed in [13, Example 35].

LEMMA 7.2. *The face maps of Definition 7.1 are well defined.*

Proof. We want that if $[\alpha]_{n-p-1} = [\eta]_{n-p-1}$, then $\partial_k^p([\alpha]_{n-p-1}) = \partial_k^p([\eta]_{n-p-1})$. If $[\alpha]_{n-p-1} = [\eta]_{n-p-1}$, then $\bar{\alpha} = \bar{\eta}$ where the bar is taken with respect to A_{n-p-1}^+ . Set $\bar{\alpha} = \gamma$ (recall the definition of $\bar{\alpha}$ from Remark 4.17). It follows that there exist a and b in A_{n-p-1}^+ such that $\alpha = \gamma a$ and $\eta = \gamma b$. Then since a and b only contain letters in A_{n-p-1}^+ and all of these letters commute with $(\sigma_{n-p+k}\sigma_{n-p+k-1} \cdots \sigma_{n-p+1})$ it follows that a and b letterwise commute with the face map. Taking equivalence classes with respect to A_{n-p}^+ therefore gives

$$\begin{aligned} &[\alpha(\sigma_{n-p+k}\sigma_{n-p+k-1} \cdots \sigma_{n-p+1})]_{n-p} \\ &= [(\gamma a)(\sigma_{n-p+k}\sigma_{n-p+k-1} \cdots \sigma_{n-p+1})]_{n-p} \\ &= [\gamma(\sigma_{n-p+k}\sigma_{n-p+k-1} \cdots \sigma_{n-p+1})a]_{n-p} \\ &= [\gamma(\sigma_{n-p+k}\sigma_{n-p+k-1} \cdots \sigma_{n-p+1})]_{n-p} \end{aligned}$$

and similarly

$$\begin{aligned} &[\eta(\sigma_{n-p+k}\sigma_{n-p+k-1} \cdots \sigma_{n-p+1})]_{n-p} \\ &= [\gamma(\sigma_{n-p+k}\sigma_{n-p+k-1} \cdots \sigma_{n-p+1})]_{n-p} \end{aligned}$$

and so the face maps are well defined. □

LEMMA 7.3. *The face maps $\{\partial_k^p\}$ on \mathcal{C}_\bullet^n defined in Definition 7.1 satisfy the simplicial identities, that is, for $0 \leq i < j \leq p$:*

$$\partial_i^{p-1} \partial_j^p = \partial_{j-1}^{p-1} \partial_i^p : \mathcal{C}_p^n \rightarrow \mathcal{C}_{p-2}^n.$$

Proof. For ease of notation in the proof, we denote $(n - p)$ as r . Then the left-hand side acts as follows

$$\begin{array}{ccccc} \mathcal{C}_p^n & \xrightarrow{\partial_j^p} & \mathcal{C}_{p-1}^n & \xrightarrow{\partial_i^{p-1}} & \mathcal{C}_{p-2}^n \\ [\alpha]_{r-1} & \xrightarrow{\partial_j^p} & [\alpha(\sigma_{r+j} \dots \sigma_{r+1})]_r & \xrightarrow{\partial_i^{p-1}} & [\alpha(\sigma_{r+j} \dots \sigma_{r+1})(\sigma_{r+i+1} \dots \sigma_{r+2})]_{r+1}. \end{array}$$

In comparison, the right-hand side acts as follows

$$\begin{array}{ccccc} \mathcal{C}_p^n & \xrightarrow{\partial_i^p} & \mathcal{C}_{p-1}^n & \xrightarrow{\partial_{j-1}^{p-1}} & \mathcal{C}_{p-2}^n \\ [\alpha]_{r-1} & \xrightarrow{\partial_i^p} & [\alpha(\sigma_{r+i} \dots \sigma_{r+1})]_r & \xrightarrow{\partial_{j-1}^{p-1}} & [\alpha(\sigma_{r+i} \dots \sigma_{r+1})(\sigma_{r+j} \dots \sigma_{r+2})]_{r+1} \end{array}$$

Let $x = (\sigma_{r+j} \dots \sigma_{r+1})(\sigma_{r+i+1} \dots \sigma_{r+2})$ and $y = (\sigma_{r+i} \dots \sigma_{r+1})(\sigma_{r+j} \dots \sigma_{r+2})$. Note that for $0 \leq k < j$ we have

$$(\sigma_{r+j} \dots \sigma_{r+1})\sigma_{r+k+1} = \sigma_{r+k}(\sigma_{r+j} \dots \sigma_{r+1})$$

from manipulation of the words using the braiding relations in the monoid. Reiterating this gives us the first equality in the following:

$$\begin{aligned} x &= (\sigma_{r+j} \dots \sigma_{r+1})(\sigma_{r+i+1} \dots \sigma_{r+2}) \\ &= (\sigma_{r+i} \dots \sigma_r)(\sigma_{r+j} \dots \sigma_{r+1})\sigma_{r+2} \\ &= (\sigma_{r+i} \dots \sigma_r)(\sigma_{r+j} \dots \sigma_{r+3})(\sigma_{r+2}\sigma_{r+1}\sigma_{r+2}) \\ &= (\sigma_{r+i} \dots \sigma_r)(\sigma_{r+j} \dots \sigma_{r+3})(\sigma_{r+1}\sigma_{r+2}\sigma_{r+1}) \\ &= (\sigma_{r+i} \dots \sigma_r)\sigma_{r+1}(\sigma_{r+j} \dots \sigma_{r+3})\sigma_{r+2}\sigma_{r+1} \\ &= (\sigma_{r+i} \dots \sigma_r\sigma_{r+1})(\sigma_{r+j} \dots \sigma_{r+3}\sigma_{r+2})\sigma_{r+1} \\ &= y\sigma_{r+1}. \end{aligned}$$

The result follows since we are taking the equivalence relation with respect to the submonoid A_{r+1}^+ . □

LEMMA 7.4. *Recall the notation $A^+(n; n - p - 1)$, as defined in Section 6. Then the realisation of the map defined levelwise on the bar construction by*

$$\begin{aligned} B_p(A_n^+, A_{n-p-1}^+, *) &\rightarrow A^+(n; n - p - 1) \\ \alpha[m_1, \dots, m_p] &\mapsto \bar{\alpha}, \end{aligned}$$

where $\alpha \in A_n^+$, $m_i \in A_{n-p-1}^+$ for all i and $\alpha = \bar{\alpha}\beta$ for $\bar{\alpha} \in A^+(n; n - p - 1)$ and $\beta \in A_{n-p-1}^+$ is a homotopy equivalence. That is the p th level of \mathcal{C}_\bullet^n satisfies

$$A_n^+ \parallel A_{n-p-1}^+ \simeq A^+(n; n - p - 1) = \mathcal{C}_p^n.$$

Proof. This is a direct application of Propositions 4.23 and 5.18 which gives the decomposition $A_n^+ \cong A^+(n; n - p - 1) \times A_{n-p-1}^+$. □

DEFINITION 7.5. Let \mathcal{A}_\bullet^n be the semi-simplicial space with p th level the homotopy quotient $\mathcal{A}_p^n = A_n^+ \parallel \mathcal{C}_p^n$, where the action of A_n^+ on $A^+(n; n - p - 1)$ is given by

$$a \cdot [\alpha]_{n-p-1} = [a\alpha]_{n-p-1} \text{ for } a, \alpha \in A_n^+.$$

The face maps are denoted by ∂_k^p for $0 \leq k \leq p$

$$\begin{aligned} \partial_k^p : \mathcal{A}_p^n &\rightarrow \mathcal{A}_{p-1}^n \\ \partial_k^p : A_n^+ \parallel \mathcal{C}_p^n &\rightarrow A_n^+ \parallel \mathcal{C}_{p-1}^n \end{aligned}$$

and ∂_k^p acts as the face map ∂_k^p from Definition 7.1 on the \mathcal{C}_p^n factor of each simplex in the homotopy quotient, and as the identity on the other factors.

Diagrammatically, \mathcal{A}_\bullet^n can be drawn as:

$$\begin{array}{ccc} A_n^+ \parallel \mathcal{C}_{n-1}^n & & \mathcal{A}_{n-1}^n \\ \downarrow \downarrow \dots \downarrow \downarrow & & \\ A_n^+ \parallel \mathcal{C}_{n-2}^n & & \mathcal{A}_{n-2}^n \\ \downarrow \downarrow \dots \downarrow \downarrow & & \\ A_n^+ \parallel \mathcal{C}_{n-3}^n & & \mathcal{A}_{n-3}^n \\ \downarrow \downarrow \dots \downarrow \downarrow & & \\ \vdots & & \vdots \\ \downarrow \downarrow \downarrow & & \\ A_n^+ \parallel \mathcal{C}_1^n & & \mathcal{A}_1^n \\ \downarrow \downarrow & & \\ A_n^+ \parallel \mathcal{C}_0^n & & \mathcal{A}_0^n \end{array}$$

LEMMA 7.6. The factorwise definition of the face maps ∂_k^p in Definition 7.5 gives well-defined maps on the homotopy quotients at each level of \mathcal{A}_\bullet^n .

Proof. The set of j -simplices in $A_n^+ \parallel \mathcal{C}_p^n$ is identified with $(A_n^+)^j \times \mathcal{C}_p^n$ and a generic element in this set is given by $[a_1, \dots, a_j][\alpha]_{n-p-1}$, where the a_i and α are in A_n^+ . Then the map ∂_k^p acts on this simplex as

$$\partial_k^p([a_1, \dots, a_j][\alpha]_{n-p-1}) \mapsto [a_1, \dots, a_j][\alpha(\sigma_{n-p+k}\sigma_{n-p+k-1} \dots \sigma_{n-p+1})]_{n-p}$$

and since the multiplication by $(\sigma_{n-p+k}\sigma_{n-p+k-1} \dots \sigma_{n-p+1})$ is on the right it follows that ∂_k^p commutes with all face maps of the bar construction $B_\bullet(*, A_n^+, \mathcal{C}_p^n)$ for each k . Therefore, the definition of ∂_k^p on the simplicial level induces a map on the homotopy quotient $A_n^+ \parallel \mathcal{C}_p^n$. \square

LEMMA 7.7. The face maps ∂_k^p on \mathcal{A}_\bullet^n defined in Definition 7.5 satisfy the simplicial identities, that is for $0 \leq i < j \leq p$:

$$\partial_i^{p-1}\partial_j^p = \partial_{j-1}^{p-1}\partial_i^p.$$

Proof. This follows directly from the fact that the simplicial identities are satisfied for \mathcal{C}_\bullet^n (Lemma 7.3), since the face maps for \mathcal{A}_\bullet^n are defined via the maps for \mathcal{C}_\bullet^n . \square

We now show that there exist homotopy equivalences $\mathcal{A}_p^n \simeq BA_{n-p-1}^+$ for every $p \geq 0$, as promised in Section 1.1.

LEMMA 7.8. Consider the levelwise maps on (j, k) -simplices of $A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+$:

$$(A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+)_{(j,k)} \rightarrow (A_n^+ \parallel \mathcal{C}_p^n)_j$$

$$[a_1, \dots, a_j] \alpha [a'_1, \dots, a'_k] \mapsto [a_1, \dots, a_j] \bar{\alpha}$$

and the projection

$$(A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+)_{(j,k)} \rightarrow (* \parallel A_{n-p-1}^+)_k$$

$$[a_1, \dots, a_j] \alpha [a'_1, \dots, a'_k] \mapsto *[a'_1, \dots, a'_k],$$

where α and $a_i \in A_n^+$, $a'_i \in A_{n-p-1}^+$, and $\alpha = \bar{\alpha}\beta$ for $\bar{\alpha} \in A^+(n; n-p-1)$ and $\beta \in A_{n-p-1}^+$. Then these maps are homotopy equivalences

$$A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+ \simeq \mathcal{A}_p^n$$

and

$$A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+ \simeq BA_{n-p-1}^+,$$

respectively. That is, the p th level of the space \mathcal{A}_\bullet^n satisfies

$$\mathcal{A}_p^n \simeq A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+ \simeq BA_{n-p-1}^+.$$

Proof. From Lemma 7.4, $\mathcal{C}_p^n = A^+(n; n-p-1) \simeq A_n^+ \parallel A_{n-p-1}^+$, and this induces

$$\mathcal{A}_p^n = A_n^+ \parallel \mathcal{C}_p^n \simeq A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+$$

with the homotopy equivalence given by the required map. We then have the following

$$\mathcal{A}_p^n \simeq A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+ = (A_n^+ \parallel A_n^+) \parallel A_{n-p-1}^+ \simeq * \parallel A_{n-p-1}^+ = BA_{n-p-1}^+.$$

The central equality is due to the fact that the double homotopy quotient is the geometric realisation of a bi-simplicial-set and therefore we can take the realisation in either direction first. The second map in the previous equation is a homotopy equivalence by Lemma 5.16. \square

We now define the map from the geometric realisation of \mathcal{A}_\bullet^n to the classifying space BA_n^+ promised in Section 1.1:

$$\|\mathcal{A}_\bullet^n\| \xrightarrow{\|\phi_\bullet\|} BA_n^+.$$

In Section 8, we will show that $\|\phi_\bullet\|$ is highly connected.

LEMMA 7.9. The geometric realisation $\|\mathcal{A}_\bullet^n\|$ satisfies $\|\mathcal{A}_\bullet^n\| \cong A_n^+ \parallel \|\mathcal{C}_\bullet^n\|$.

Proof. The face maps in the bar construction $B_\bullet(*, A_n^+, \mathcal{C}_p^n)$ for the homotopy quotient in $\mathcal{A}_p^n = A_n^+ \parallel \mathcal{C}_p^n$ commute with the face maps in \mathcal{C}_\bullet^n (see the proof of Lemma 7.6) and therefore with the face maps of \mathcal{A}_\bullet^n . Therefore, the two simplicial directions create a bi-semi-simplicial space and one can realise in either direction first, as in Definition 5.11. Realising by taking the homotopy quotients $\mathcal{A}_p^n = A_n^+ \parallel \mathcal{C}_p^n$ before realising in the \mathcal{A}_\bullet^n direction first (which has face maps induced by those of \mathcal{C}_\bullet^n) gives the left-hand side. Realising in the \mathcal{C}_\bullet^n direction before taking the homotopy quotient $A_n^+ \parallel \|\mathcal{C}_\bullet^n\|$ gives the right-hand side. \square

Recall that $A_n^+ \parallel *$ is a model for BA_n^+ . We therefore define $\|\phi_\bullet\|$ as a map from $A_n^+ \parallel \|\mathcal{C}_\bullet^n\|$ to $A_n^+ \parallel *$.

DEFINITION 7.10. Define ϕ_\bullet to be the semi-simplicial map from the bar construction $B_\bullet(*, A_n^+, \|\mathcal{C}_\bullet^n\|)$ to the bar construction $B_\bullet(*, A_n^+, *)$, obtained by collapsing $\|\mathcal{C}_\bullet^n\|$ to a point:

$$\begin{aligned} \phi_p : B_p(*, A_n^+, \|\mathcal{C}_\bullet^n\|) &\rightarrow B_p(*, A_n^+, *) \\ [a_1, \dots, a_p]a &\mapsto [a_1, \dots, a_p]*, \end{aligned}$$

where a_i is in A_n^+ for all i , and a is in $\|\mathcal{C}_\bullet^n\|$. Then the geometric realisation $\|\phi_\bullet\|$ maps the homotopy quotient $A_n^+ \amalg \|\mathcal{C}_\bullet^n\|$ to the homotopy quotient $A_n^+ \amalg * \simeq BA_n^+$.

PROPOSITION 7.11. *If $\|\mathcal{C}_\bullet^n\|$ is $(k - 1)$ -connected, then the map $\|\phi_\bullet\|$ is k -connected.*

Proof. From [12, Lemma 2.4], a semi-simplicial map $f_\bullet : X_\bullet \rightarrow Y_\bullet$ satisfies that $\|f_\bullet\|$ is k -connected if $f_p : X_p \rightarrow Y_p$ is $(k - p)$ connected for all $p \geq 0$. The map $\|\phi_\bullet\|$ is defined levelwise as the projection

$$\phi_p : (A_n^+)^p \times \|\mathcal{C}_\bullet^n\| \rightarrow (A_n^+)^p.$$

Therefore, since $\|\mathcal{C}_\bullet^n\|$ is $(k - 1)$ -connected it follows that ϕ_p is k -connected and in particular it is $(k - p)$ -connected for all $p \geq 0$. Thus, the geometric realisation $\|\phi_\bullet\|$ is k -connected. \square

8. High connectivity

This section is concerned with the proof of the following theorem.

THEOREM 8.1. *The geometric realisation $\|\mathcal{C}_\bullet^n\|$ of the semi-simplicial space \mathcal{C}_\bullet^n is $(n - 2)$ -connected for all n , that is, $\pi_i(\|\mathcal{C}_\bullet^n\|) = 0$ for $0 \leq i \leq n - 2$.*

Combining this theorem with Proposition 7.11, it follows that the map $\|\phi_\bullet\|$ is $(n - 1)$ -connected as promised in Section 1.1. For the remainder of this paper, we will refer to the geometric realisation of the semi-simplicial space as a *complex* (the geometric realisation is, by definition, a cell complex: note that it is not necessarily a simplicial complex).

8.1. Union of chambers argument

There is a specific argument, called a *union of chambers argument* that is often used to prove high connectivity of a complex. It is closely related to the notion of *shellability*.

In [8], Davis used a union of chambers argument to prove that the Davis complex Σ_W associated to a Coxeter group is contractible. He did this by showing that the Davis complex is an example of a so-called *basic construction*. Hepworth’s high-connectivity results relating to homological stability for Coxeter groups [13] also used such an argument. In [19], Paris used a union of chambers argument to show that the universal cover of an analogue of the Salvetti complex for certain Artin monoids is contractible, proving the $K(\pi, 1)$ conjecture for finite-type Artin groups. In this chapter, we use a similar union of chambers argument to prove high connectivity. Loosely, the argument consists of breaking the complex up into high-dimensional *chambers* and considering how connectivity changes as they are glued together to create the complex. While applying the argument in the case of Artin monoids and the complex we have constructed, numerous technical challenges arise, leading to the proof being split into many separate cases.

To prove high connectivity in our setup, we use a union of chambers argument applied to the complex $\|\mathcal{C}_\bullet^n\|$. Recall that $\|\mathcal{C}_\bullet^n\|$ has dimension $n - 1$. We filter the top dimensional simplices by the natural numbers as follows.

DEFINITION 8.2. For k in \mathbb{N} we define $\mathcal{C}^n(k)$ as follows:

$$\mathcal{C}^n(k) = \bigcup_{\substack{\alpha \in A_n^+, \\ \ell(\alpha) \leq k}} \llbracket \alpha \rrbracket_0,$$

where $\llbracket \alpha \rrbracket_0$ is the $(n - 1)$ simplex in $\|\mathcal{C}_\bullet^n\|$ represented by $[\alpha]_0$ in \mathcal{C}_{n-1}^n .

REMARK 8.3. Note that every simplex in $\|\mathcal{C}_\bullet^n\|$ arises as a face of some $\llbracket \alpha \rrbracket_0$, since smaller simplices are represented by some $\llbracket \tau \rrbracket_k$ for $k > 0$ and this is a face of $\llbracket \tau \rrbracket_0$. Then $\|\mathcal{C}_\bullet^n\|$ is given by $\text{colim}_{k \rightarrow \infty} \mathcal{C}^n(k)$.

The union of chambers argument relies on the following two steps.

- (A) If $\ell(\alpha) = k + 1$, then $\llbracket \alpha \rrbracket_0 \cap \mathcal{C}^n(k)$ is a non-empty union of top dimensional faces of $\llbracket \alpha \rrbracket_0$.
- (B) If $\ell(\alpha) = \ell(\beta) = k + 1$ and $\alpha \neq \beta$, then $\llbracket \alpha \rrbracket_0 \cap \llbracket \beta \rrbracket_0 \subseteq \mathcal{C}^n(k)$.

PROPOSITION 8.4. If (A) and (B) hold, then $\|\mathcal{C}_\bullet^n\|$ is homotopy equivalent to a wedge of $(n - 1)$ spheres, and in particular is $(n - 2)$ connected.

Proof. We build up $\|\mathcal{C}_\bullet^n\|$ by increasing k in $\mathcal{C}^n(k)$. We start at $\mathcal{C}^n(0) = \llbracket e \rrbracket_0$, which is contractible. At each step we build up from $\mathcal{C}^n(k)$ to $\mathcal{C}^n(k + 1)$ by adding the set of simplices represented by words in A_n^+ of length $(k + 1)$:

$$\bigcup_{\substack{\alpha \in A_n^+, \\ \ell(\alpha) = k + 1}} \llbracket \alpha \rrbracket_0.$$

Then point (A) says that when $\llbracket \alpha \rrbracket_0$ is added to $\mathcal{C}^n(k)$, the intersection is a non-empty union of facets of $\llbracket \alpha \rrbracket_0$. Therefore, either the homotopy type doesn't change upon adding the simplex (if not all facets are in the intersection), or the homotopy changes and this change is described by the possible addition of an $(n - 1)$ sphere (if all facets are in the intersection). Point (B) then says that adding the entirety of the above union to $\mathcal{C}^n(k)$ at the same time only changes the homotopy type in the sense that the individual simplices change it, since each two simplices intersect within $\mathcal{C}^n(k)$. Therefore, at each stage we change the homotopy type by at most the addition of several $(n - 1)$ spheres and it follows that $\|\mathcal{C}_\bullet^n\|$ is $(n - 2)$ connected. \square

REMARK 8.5. The length function gives a partial order on the top dimensional simplices of $\|\mathcal{C}_\bullet^n\|$. By (B), any linear extension of this partial order to a total order will still satisfy (A). In this case, the ordering is called a *shelling* (see [1]), which we know to be highly connected: giving an alternative proof to the previous proposition.

The remainder of this section is therefore devoted to the proof of point (A) and point (B). The proof of point (A) is split into several subsections.

8.2. Proof of point (A): facets of $\llbracket \alpha \rrbracket_0$

Recall that the top dimensional faces of a simplex are called *facets*. We start the proof of point (A) with a discussion of the facets of a simplex $\llbracket \alpha \rrbracket_0$. Consider the face maps

$$\begin{aligned} \partial_q^{n-1} : \mathcal{C}_{n-1}^n &\rightarrow \mathcal{C}_{n-2}^n \\ \partial_q^{n-1} : \llbracket \alpha \rrbracket_0 &\mapsto \llbracket \alpha \sigma_{q+1} \sigma_q \dots \sigma_2 \rrbracket_1 \end{aligned}$$

for $0 \leq q \leq n - 1$. The map ∂_0^{n-1} is right multiplication by the identity.

Under these face maps the facets of $[[\alpha]]_0$ are given by

$$[[\alpha]]_1, [[\alpha\sigma_2]]_1, [[\alpha\sigma_3\sigma_2]]_1, [[\alpha\sigma_4\sigma_3\sigma_2]]_1, \dots, [[\alpha\sigma_n\sigma_{n-1} \dots \sigma_3\sigma_2]]_1$$

PROPOSITION 8.6. *If $\ell(\alpha) = k + 1$, at least one of the facets of $[[\alpha]]_0$ lies in $\mathcal{C}^n(k)$.*

Proof. We must show that at least one facet of $[[\alpha]]_0$ is also a facet of some simplex $[[\alpha']]_0$, where $\ell(\alpha') \leq k$.

Consider $\text{EndGen}_1(\alpha)$. If this is non-empty, then there exists η with length at least 1 in A_1^+ such that $\alpha = \alpha'\eta$. It follows that $[[\alpha]]_1 = [[\alpha'\eta]]_1 = [[\alpha']]_1$. Therefore, the facet $[[\alpha]]_1$ is also a facet of $[[\alpha']]_0$. Since $\ell(\eta) \geq 1$, then $\ell(\alpha') < \ell(\alpha) = k + 1$, so $[[\alpha']]_0$ is in $\mathcal{C}^n(k)$.

Alternatively if $\text{EndGen}_1(\alpha) = \emptyset$, then $\ell(\alpha) \geq 1$ implies that $\text{EndGen}_n(\alpha) \neq \emptyset$. It follows from these two observations that $\{\sigma_2, \dots, \sigma_n\} \cap \text{EndGen}_n(\alpha) \neq \emptyset$, that is, for some $2 \leq j \leq n$, $\alpha = \alpha'\sigma_j$. Applying the face map ∂_{j-2}^n gives

$$\begin{aligned} \partial_{j-2}^{n-1}([[\alpha]]_0) &= [[\alpha\sigma_{j-1} \dots \sigma_2]]_1 \\ &= [[\alpha'\sigma_j\sigma_{j-1} \dots \sigma_2]]_1 \\ &= \partial_{j-1}^{n-1}([[\alpha']]_0) \end{aligned}$$

and as before $\ell(\alpha') \leq k$. This shows that the facet $\partial_{j-2}^{n-1}([[\alpha]]_0)$ is also a facet of $[[\alpha']]_0$ and is therefore in $\mathcal{C}^n(k)$. □

To complete the proof of point (A) we must show that if a lower dimensional face of $[[\alpha]]_0$ is contained in $\mathcal{C}^n(k)$, then it is contained in a facet of $[[\alpha]]_0$, which is itself contained in $\mathcal{C}^n(k)$. We first describe a general form for faces of $[[\alpha]]_0$.

8.3. Proof of point (A): low-dimensional faces of $[[\alpha]]_0$

DEFINITION 8.7. A face of $[[\alpha]]_0$ is obtained by applying a series of face maps to $[[\alpha]]_0$. We denote the series of face maps applied by a tuple $(\partial_{i_2}^{n-1}, \partial_{i_3}^{n-2}, \dots, \partial_{i_r}^{n-r+1})$, and we let $a_j := \sigma_{i_{j-1}+j} \dots \sigma_j$. That is, the $(j - 1)$ st map in the tuple corresponds to right multiplication by a_j . We note here that a_j has length i_j and ends with the generator σ_j , unless $i_j = 0$ in which case $a_j = e$.

$$\begin{aligned} \partial_{i_j}^{n-j+1} : \mathcal{C}_{n-j+1}^n &\rightarrow \mathcal{C}_{n-j}^n \\ &: [[\alpha]]_{j-2} \mapsto [[\alpha\sigma_{i_{j-1}+j} \dots \sigma_j]]_{j-1} \\ &= [[\alpha a_j]]_{j-1}. \end{aligned}$$

From now on we assume that the first map in a tuple maps from \mathcal{C}_{n-1}^n to \mathcal{C}_{n-2}^n , the second map from \mathcal{C}_{n-2}^n to \mathcal{C}_{n-3}^n and so on. We therefore dispense of the superscripts in the ∂ notation for the face maps when we write these tuples.

With the above notation, an $(n - p - 1)$ subsimplex of $[[\alpha]]_0$ occurs when a tuple of face maps $(\partial_{i_2}, \partial_{i_3}, \dots, \partial_{i_{p+1}})$ is applied to $[[\alpha]]_0$. The image of these maps is then the subsimplex $[[\alpha a_2 \dots a_{p+1}]]_p$ with a_j defined as in Definition 8.7 above.

LEMMA 8.8. *With the above notation, the tuple of face maps $(\partial_{i_j})_{j=2}^{p+1}$ can be written such that $i_{j+1} \geq i_j$ for all j , which translates to $\ell(a_{j+1}) \geq \ell(a_j)$.*

Proof. This is a direct consequence of Lemma 5.6. □

LEMMA 8.9. *The $(n - p - 1)$ subsimplex of $[[\alpha]]_0$ given by $(\partial_{i_2}, \partial_{i_3}, \dots, \partial_{i_{p+1}})$, or alternatively $[[\alpha a_2 \dots a_{p+1}]]_p$, is a subsimplex of the following facets of $[[\alpha]]_0$.*

- $\partial_{i_2}([\alpha]_0) = [[\alpha a_2]]_1$.
- $\partial_{i_3+1}([\alpha]_0) = [[\alpha a_3 \sigma_2]]_1$.
- $\partial_{i_4+2}([\alpha]_0) = [[\alpha a_4 \sigma_3 \sigma_2]]_1$.
- \dots
- $\partial_{i_{p+1}+p-1}([\alpha]_0) = [[\alpha a_{p+1} \sigma_p \dots \sigma_2]]_1$.

In general these facets are given by the face map

$$\partial_{i_j+(j-2)} : [[\alpha]]_0 \mapsto [[\alpha a_j \sigma_{j-1} \dots \sigma_2]]_1.$$

Proof. It is enough to show that $\partial_{i_j+(j-2)}$ can act as the first face map in the tuple $(\partial_{i_2}, \partial_{i_3}, \dots, \partial_{i_{p+1}})$ for all j . Recall from Lemma 8.8 that in the tuple $i_{j+1} \geq i_j$ for all j . Using the simplicial identities, the tuple is equivalent to the tuple $(\partial_{i_j+(j-2)}, \partial_{i_2}, \partial_{i_3}, \dots, \widehat{\partial_{i_j}}, \dots, \partial_{i_{p+1}})$. \square

For the remainder of this section, let α in A_n^+ with $\ell(\alpha) = k + 1$. The aim of this section is to show that if the $(n - p - 1)$ subsimplex of $[[\alpha]]_0$ given by $(\partial_{i_2}, \partial_{i_3}, \dots, \partial_{i_{p+1}})$ or alternatively $[[\alpha a_2 \dots a_{p+1}]]_p$ is in $\mathcal{C}^n(k)$, then it follows that one of the facets of $[[\alpha]]_0$ from Lemma 8.9 is also in $\mathcal{C}^n(k)$. The proof of point (A) will follow.

DEFINITION 8.10. If $[[\alpha a_2 \dots a_{p+1}]]_p$ is in $\mathcal{C}^n(k)$, then it is also a $(n - p - 1)$ subsimplex of a simplex $[[\beta]]_0$ for some β in A_n^+ such that $\ell(\beta) \leq k$. The subsimplex is therefore obtained from $[[\beta]]_0$ by applying a tuple of face maps, denote these $(\partial_{l_2}, \partial_{l_3}, \dots, \partial_{l_{p+1}})$ and order as in Lemma 8.8 such that $l_{j+1} \geq l_j$ for all j . Define $b_j := \sigma_{l_j-1+j} \dots \sigma_j$ and when $l_j = 0$ let $b_j = e$. Then $(\partial_{l_2}, \partial_{l_3}, \dots, \partial_{l_{p+1}})$ applied to $[[\beta]]_0$ gives the $(n - p - 1)$ simplex $[[\beta b_2 \dots b_{p+1}]]_p$. By construction $[[\beta b_2 \dots b_{p+1}]]_p = [[\alpha a_2 \dots a_{p+1}]]_p$. We recall here that $\ell(a_j) = i_j$ and $\ell(b_j) = l_j$.

LEMMA 8.11. *Choose β and b_j as defined above, such that $\sum_{k=2}^{p+1} l_k$ is minimal, corresponding to $b_2 \dots b_{p+1}$ being of minimal length. This choice of $b_2 \dots b_{p+1}$ then corresponds to either:*

$$[[\alpha a_2 \dots a_{p+1}]]_p = [[\beta]]_p \text{ that is, } l_j = 0 \forall j$$

or

$$\ell(\beta) = \ell(\alpha) - 1 = k.$$

Proof. Suppose that β and b_j are chosen such that $\sum_{k=2}^{p+1} l_k$ is minimal, and furthermore suppose that $\ell(\beta) < \ell(\alpha) - 1$ and $\sum_{k=2}^{p+1} l_k > 0$. Then some $l_k \neq 0$: set j to be minimal such that $l_j \neq 0$. Then $b_j = \sigma_{l_j-1+j} \dots \sigma_j \neq e$ and

$$[[[\beta b_2 \dots b_{p+1}]]_p = [[[\beta b_j \dots b_{p+1}]]_p = [[[\beta \sigma_{l_j-1+j} \dots \sigma_j b_{j+1} \dots b_{p+1}]]_p.$$

But this is the tuple of face maps $(\partial_{l_j-1}, \partial_{l_{j+1}}, \dots, \partial_{l_{p+1}})$ applied to $[[[\beta \sigma_{l_j-1+j}]]_0$. Since $\ell(\beta) < \ell(\alpha) - 1$ it follows that $\ell(\beta \sigma_{l_j-1+j}) \leq \ell(\alpha) - 1$ and so $[[[\beta \sigma_{l_j-1+j}]]_0$ is in $\mathcal{C}^n(k)$. However, the tuple for $\beta \sigma_{l_j-1+j}$ has the sum of its corresponding l_j less than the original tuple for β . This is a contradiction, as β was chosen to have minimal $\sum_{k=2}^{p+1} l_k$. Therefore, either $\sum_{k=2}^{p+1} l_k = 0$, or $\ell(\beta) = \ell(\alpha) - 1$. \square

For the remainder of this paper, assume β and b_j are chosen such that $\sum_{k=2}^{p+1} l_k$ is minimal, so we have

$$[[[\beta b_2 \dots b_{p+1}]]_p = [[[\alpha a_2 \dots a_{p+1}]]_p$$

for either $\sum_{k=2}^{p+1} l_k = 0$ or $\ell(\beta) = \ell(\alpha) - 1 = k$. We use the following notation throughout the remainder of this paper.

DEFINITION 8.12. Let $a := a_2 \dots a_{p+1}$ and $b := b_2 \dots b_{p+1}$. Note that $\sum_{k=2}^{p+1} l_k = 0$ corresponds to $b = e$. So, we have

$$[[\alpha a]]_p = [[\beta b]]_p,$$

where either $b = e$, or $\ell(\beta) = \ell(\alpha) - 1$. We recall that this is equivalent to $\overline{\alpha a} = \overline{\beta b}$ in $A^+(n; p)$. Let $\gamma := \overline{\alpha a} = \overline{\beta b}$, and define u and v in A_p^+ such that

$$\alpha a = \gamma u \text{ and } \beta b = \gamma v.$$

Recall point A: if $\ell(\alpha) = k + 1$, then $[[\alpha]]_0 \cap \mathcal{C}^n(k)$ is a non-empty union of top dimensional faces of $[[\alpha]]_0$. Recall that we have fixed a face $[[\alpha a]]_p$ of $[[\alpha]]_0$, and we wish to prove this is contained in a facet of $[[\alpha]]_0$ (from Lemma 8.9) which is contained in $\mathcal{C}^n(k)$. We complete the proof of this by splitting into three cases:

- (i) $\ell(\beta b) < \ell(\alpha a)$,
- (ii) $\ell(\beta b) = \ell(\alpha a)$,
- (iii) $\ell(\beta b) > \ell(\alpha a)$,

and since multiplication in the Artin monoid corresponds to adding lengths the conditions of these cases correspond to analogous conditions on the lengths of u and v .

REMARK 8.13. Note that if $\sum_{k=2}^{p+1} l_k = 0$, then $b = e$, and since $\ell(\beta) < \ell(\alpha)$ it follows we are therefore in case (i): $\ell(\beta b) < \ell(\alpha a)$.

We prove the three cases one by one in the following subsections. This involves some technical lemmas, and in particular computation of least common multiples of strings of words. We therefore include these technical lemmas in a separate section and refer to them as required.

8.4. Proof of point (A): preliminary lemmas

Recall from Definition 8.7 that a face of $[[\alpha]]_0$ is obtained by applying a series of face maps to $[[\alpha]]_0$. We denote the series of face maps by a tuple $(\partial_{i_2}^{n-1}, \partial_{i_3}^{n-2}, \dots, \partial_{i_r}^{n-r+1})$, and we let $a_j = \sigma_{i_j-1+j} \dots \sigma_j$ and when $i_j = 0$ let $a_j = e$. That is, the $(j - 1)$ st map in the tuple corresponds to right multiplication by a_j . We let $a = a_2 \dots a_{p+1}$. Recall also that if $[[\alpha a]]_p$ is in $\mathcal{C}^n(k)$, then the subsimplex is also obtained from some $[[\beta]]_0$ for $\ell(\beta) \leq k$, by applying a tuple of face maps $(\partial_{l_2}, \partial_{l_3}, \dots, \partial_{l_{p+1}})$. Recall $b_j := \sigma_{l_j-1+j} \dots \sigma_j$ and when $l_j = 0$ let $b_j = e$. Let $b = b_2 \dots b_{p+1}$. By construction $[[\beta b]]_p = [[\alpha a]]_p$. Recall from Definition 4.9 that for α and β two words in A^+ , we denote the least common multiple of α and β (if it exists) by $\Delta(\alpha, \beta)$.

LEMMA 8.14. For all $k > j$, the generators σ_i satisfy

$$(\sigma_k \dots \sigma_{j+1})\sigma_j(\sigma_k \dots \sigma_{j+1}) = (\sigma_{k-1}\sigma_k)(\sigma_{k-2}\sigma_{k-1}) \dots (\sigma_{j+1}\sigma_{j+2})(\sigma_j\sigma_{j+1}\sigma_j).$$

Proof. We proceed by induction on $k - j$. For the base case let $k - j = 1$, that is, $k = j + 1$. Then the left-hand side of the above equation evaluates to $\sigma_{j+1}\sigma_j\sigma_{j+1}$ and the right-hand side evaluates to $\sigma_j\sigma_{j+1}\sigma_j$. These are equal by the Artin relations. For the inductive hypothesis, we assume the Lemma is true for $k - j < r$, and we prove for $k - j = r$, that is, $k = j + r$. We manipulate the left-hand side of the equation, and show equality to the right-hand side:

$$\begin{aligned}
& (\sigma_k \dots \sigma_{j+1})\sigma_j(\sigma_k \dots \sigma_{j+1}) \\
&= (\sigma_{j+r} \dots \sigma_{j+1})\sigma_j(\sigma_{j+r} \dots \sigma_{j+1}) \\
&= (\sigma_{j+r}\sigma_{j+r-1}\sigma_{j+r} \dots \sigma_{j+1})\sigma_j(\sigma_{j+r-1} \dots \sigma_{j+1}) \\
&= (\sigma_{j+r}\sigma_{j+r-1}\sigma_{j+r}) (\sigma_{j+r-2} \dots \sigma_{j+1})\sigma_j(\sigma_{j+r-1} \dots \sigma_{j+1}) \\
&= (\sigma_{j+r-1}\sigma_{j+r}\sigma_{j+r-1}) (\sigma_{j+r-2} \dots \sigma_{j+1})\sigma_j(\sigma_{j+r-1} \dots \sigma_{j+1}) \\
&= (\sigma_{j+r-1}\sigma_{j+r}) (\sigma_{j+r-1}\sigma_{j+r-2} \dots \sigma_{j+1})\sigma_j(\sigma_{j+r-1} \dots \sigma_{j+1}) \\
&= (\sigma_{j+r-1}\sigma_{j+r}) (\sigma_{j+r-2}\sigma_{j+r-1}) \dots (\sigma_{j+1}\sigma_{j+2}) (\sigma_j\sigma_{j+1}\sigma_j),
\end{aligned}$$

where the final equality applies the inductive hypothesis. \square

LEMMA 8.15. *With notation as above, $\Delta(a_{j+1}, \sigma_j) = a_{j+1}\sigma_j a_{j+1}$ (and similarly $\Delta(b_{j+1}, \sigma_j) = b_{j+1}\sigma_j b_{j+1}$).*

Proof. The proof is the same for both the a_j and b_j case, so we prove it for the a_j case. We must show:

- (a) $a_{j+1} \preceq_R a_{j+1}\sigma_j a_{j+1}$ and $\sigma_j \preceq_R a_{j+1}\sigma_j a_{j+1}$;
- (b) if x in A_n^+ is a common multiple of a_{j+1} and σ_j , then $a_{j+1}\sigma_j a_{j+1} \preceq_R x$.

Recall $a_{j+1} := \sigma_{i_{j+1}+j} \dots \sigma_{j+1}$. Without loss of generality, we relabel $j = 1$ and $i_{j+1} + j = k$. Then $a_{j+1} = \sigma_k \dots \sigma_2$ and $\sigma_j = \sigma_1$.

To prove (a) note that $a_{j+1} \preceq_R a_{j+1}\sigma_j a_{j+1}$ by observation, and also

$$\begin{aligned}
a_{j+1}\sigma_j a_{j+1} &= (\sigma_k \dots \sigma_2)\sigma_1(\sigma_k \dots \sigma_2) \\
&= (\sigma_{k-1}\sigma_k\sigma_{k-2}\sigma_{k-1} \dots \sigma_2\sigma_3)(\sigma_1\sigma_2\sigma_1)
\end{aligned}$$

by Lemma 8.14, so $\sigma_1 = \sigma_j \preceq_R a_{j+1}\sigma_j a_{j+1}$.

To prove (b) we show by induction on $\ell(a_{j+1})$ that any common multiple x must satisfy $a_{j+1}\sigma_j a_{j+1} \preceq_R x$. When $\ell(a_{j+1}) = 1$, $a_{j+1} = \sigma_2$ and we have

$$\Delta(\sigma_2, \sigma_1) = \sigma_2\sigma_1\sigma_2 = a_{j+1}\sigma_j a_{j+1}.$$

For $\ell(a_{j+1}) = r - 1$ when $r \geq 2$, assume that $\Delta(a_{j+1}, \sigma_j) = a_{j+1}\sigma_j a_{j+1}$ and prove for $\ell(a_{j+1}) = r$. Assume x satisfies $a_{j+1} \preceq_R x$ and $\sigma_j \preceq_R x$. Since $\ell(a_{j+1}) = r$, $a_{j+1} = \sigma_{r+1} \dots \sigma_2$ and so $\sigma_{r+1} \dots \sigma_2 \preceq_R x$ which in particular gives $\sigma_r \dots \sigma_2 \preceq_R x$. By the inductive hypothesis it follows that

$$\Delta(\sigma_r \dots \sigma_2, \sigma_1) = (\sigma_r \dots \sigma_2)\sigma_1(\sigma_r \dots \sigma_2).$$

and this is in $\text{EndMon}_n(x)$ by Lemma 4.25. Let $x = x'(\sigma_r \dots \sigma_2)\sigma_1(\sigma_r \dots \sigma_2)$. Then since $\sigma_{r+1} \dots \sigma_2 \preceq_R x$, by cancellation of $\sigma_r \dots \sigma_2$ it follows that

$$\sigma_{r+1} \preceq_R x'(\sigma_r \dots \sigma_2)\sigma_1 = x'\sigma_r(\sigma_{r-1} \dots \sigma_2\sigma_1).$$

Since σ_{r+1} letterwise commutes with $(\sigma_{r-1} \dots \sigma_2\sigma_1)$, from Lemma 4.28 we have $\sigma_{r+1} \preceq_R x'\sigma_r$. From Lemma 4.25, it follows $\Delta(\sigma_{r+1}, \sigma_r) = \sigma_r\sigma_{r+1}\sigma_r \preceq_R x'\sigma_r$. By cancellation of σ_r this gives $x' = x''\sigma_r\sigma_{r+1}$, so

$$\begin{aligned}
x &= (x')(\sigma_r \dots \sigma_2)\sigma_1(\sigma_r \dots \sigma_2) \\
&= (x''\sigma_r\sigma_{r+1})(\sigma_r \dots \sigma_2)\sigma_1(\sigma_r \dots \sigma_2) \\
&= x''(\sigma_r\sigma_{r+1}\sigma_r)(\sigma_{r-1} \dots \sigma_2)\sigma_1(\sigma_r \dots \sigma_2)
\end{aligned}$$

$$\begin{aligned}
 &= x''(\sigma_{r+1}\sigma_r\sigma_{r+1})(\sigma_{r-1}\dots\sigma_2)\sigma_1(\sigma_r\dots\sigma_2) \\
 &= x''(\sigma_{r+1}\sigma_r\sigma_{r+1}\sigma_{r-1}\dots\sigma_2)\sigma_1(\sigma_r\dots\sigma_2) \\
 &= x''(\sigma_{r+1}\sigma_r\sigma_{r-1}\dots\sigma_2)\sigma_1(\sigma_{r+1}\sigma_r\dots\sigma_2) \\
 &= x''a_{j+1}\sigma_ja_{j+1},
 \end{aligned}$$

as required. □

LEMMA 8.16. *Recall from Lemma 8.15 that $\Delta(a_{j+1}, \sigma_j) = a_{j+1}\sigma_ja_{j+1}$. Then when $j \geq 2$ this expression satisfies*

$$a_{j+1}\sigma_ja_{j+1} = \hat{a}_ja_ja_{j+1}\sigma_j,$$

where $\hat{a}_j = \sigma_{i_{j+1}+j-1}\dots\sigma_{i_j+j}$ and letterwise commutes with $a_2\dots a_{j-1}$.

When $j = 1$, the expression satisfies

$$a_2\sigma_1a_2 = \hat{a}_1\sigma_1a_2\sigma_1,$$

that is, the same equality holds, setting $a_1 := \sigma_1$. The analogous statements hold for the b_j .

Proof. Recall $a_{j+1} = \sigma_{i_{j+1}+j}\dots\sigma_{j+1}$ and $a_j = \sigma_{i_j-1+j}\dots\sigma_j$. Without loss of generality, relabel $j = 1$, $i_{j+1} + j = k$, and $i_j - 1 + j = l$. Then $a_{j+1} = \sigma_k\dots\sigma_2$, $\sigma_j = \sigma_1$, and $a_j = \sigma_l\dots\sigma_1$. Note that since $i_{j+1} \geq i_j$, then $k > l$. We wish to show that $a_{j+1}\sigma_ja_{j+1} = \hat{a}_ja_ja_{j+1}\sigma_j$, where $\hat{a}_j = \sigma_{k-1}\dots\sigma_{l+1}$. We proceed by induction on the length of a_{j+1} . For the base case, when $\ell(a_{j+1}) = 1$ this implies that $a_{j+1} = \sigma_2$. Then we have

$$a_{j+1}\sigma_ja_{j+1} = \sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1.$$

Since $0 \leq i_j \leq i_{j+1} = 1$ there are now two options. In the case $i_j = 1$, the right-hand side is $a_ja_{j+1}\sigma_j$ and $\hat{a}_j = e$. In the case $i_j = 0$, then $a_j = e$ and the right-hand side is $\hat{a}_ja_{j+1}\sigma_j$ with $\hat{a}_j = \sigma_1 = \sigma_{i_j+j}$.

For the inductive hypothesis we assume true for $\ell(a_{j+1}) \leq r - 1$ and prove for $\ell(a_{j+1}) = r - 1$, that is, $k = r$. Recall from Lemma 8.14 that

$$\begin{aligned}
 a_{j+1}\sigma_ja_{j+1} &= (\sigma_k\dots\sigma_2)\sigma_1(\sigma_k\dots\sigma_2) \\
 &= (\sigma_r\dots\sigma_2)\sigma_1(\sigma_r\dots\sigma_2) \\
 &= (\sigma_{r-1}\sigma_r)(\sigma_{r-2}\sigma_{r-1})\dots(\sigma_2\sigma_3)(\sigma_1\sigma_2\sigma_1) \\
 &= (\sigma_{r-1}\sigma_r)(\sigma_{r-2}\dots\sigma_{l+1})a_j(\sigma_{r-1}\dots\sigma_2)\sigma_j,
 \end{aligned}$$

where the final equality applies the inductive hypothesis. Then σ_r commutes with $(\sigma_{r-2}\dots\sigma_{l+1})a_j$ since $\ell(a_j) \leq \ell(a_{j+1})$. This gives the following:

$$\begin{aligned}
 a_{j+1}\sigma_ja_{j+1} &= (\sigma_{r-1}\sigma_r)(\sigma_{r-2}\dots\sigma_{l+1})a_j(\sigma_{r-1}\dots\sigma_2)\sigma_j \\
 &= (\sigma_{r-1}\sigma_{r-2}\dots\sigma_{l+1})a_j(\sigma_r\sigma_{r-1}\dots\sigma_2)\sigma_j \\
 &= \hat{a}_ja_ja_{j+1}\sigma_j.
 \end{aligned}$$

Since $i_j \geq i_{j-1}$ it follows that $l - 1$ is the maximal index of a generator appearing in a_{j-1} and hence in the string $a_2\dots a_{j-1}$. Therefore, \hat{a}_j letterwise commutes with $a_2\dots a_{j-1}$ since the indices of the generators in each word pairwise differ by at least two.

Since the b_j have the same form as the a_j , with difference only in word length, the analogous statements hold for the b_j . □

Recall the definition of a_j and b_j for $2 \leq j \leq p + 1$, from Definitions 8.7 and 8.10, respectively.

DEFINITION 8.17. For $2 \leq j \leq p+1$ define c_j as follows

$$c_j = \begin{cases} a_j & \text{if } \ell(a_j) \geq \ell(b_j) \\ b_j & \text{if } \ell(a_j) < \ell(b_j) \end{cases}$$

for $2 \leq j \leq p+1$. Define $c := c_2 \dots c_{p+1}$. Let

$$a'_j = \begin{cases} e & \text{if } \ell(a_j) \geq \ell(b_j) \\ \sigma_{l_j+j-1} \dots \sigma_{i_j+j} & \text{if } \ell(a_j) < \ell(b_j) \end{cases}$$

and similarly

$$b'_j = \begin{cases} e & \text{if } \ell(b_j) \geq \ell(a_j) \\ \sigma_{i_j+j-1} \dots \sigma_{l_j+j} & \text{if } \ell(b_j) < \ell(a_j). \end{cases}$$

Define $a' = a'_2 \dots a'_{p+1}$ and $b' = b'_2 \dots b'_{p+1}$.

LEMMA 8.18. With c, a' and b' as in Definition 8.17 and a and b as defined in Definition 8.12, we have $c = \Delta(a, b)$ and in particular $c = a'a = b'b$.

Proof. We prove that:

- (a) $c = a'a = b'b$;
- (b) if x in A_n^+ is a common multiple of a and b , then $c \preceq_R x$.

To prove (a), we show that $c = a'a$: the proof that $c = b'b$ is symmetric. It follows from the definitions that $c_j = a'_j a_j$. The smallest generator index in a'_j is $(i_j + j)$ and the largest generator index in $a_2 \dots a_{j-1}$ is $(i_{j-1} + (j-1) - 1)$. The elements a'_j and $a_2 \dots a_{j-1}$ letterwise commute, since $i_j \geq i_{j-1}$ so

$$|(i_j + j) - (i_{j-1} + (j-1) - 1)| = |(i_j - i_{j-1}) + 2| \geq 2.$$

Let $a' = a'_2 \dots a'_{p+1}$. Then we compute

$$\begin{aligned} c &= c_2 \dots c_{p+1} \\ &= (a'_2 a_2)(a'_3 a_3) \dots (a'_{p+1} a_{p+1}) \\ &= a'_2 a'_3 a_2 a_3 \dots (a'_{p+1} a_{p+1}) \\ &= a'_2 a'_3 \dots a'_{p+1} a_2 a_3 \dots a_{p+1} \\ &= (a'_2 a'_3 \dots a'_{p+1})(a_2 a_3 \dots a_{p+1}) \\ &= a'a \end{aligned}$$

which completes the proof of (a).

To prove (b), assume x is a common multiple of a and b .

CLAIM. If $c_k \dots c_{p+1} \preceq_R x$ for some $2 \leq k \leq p+1$, then $x = x_k c_k \dots c_{p+1}$ for some x_k in A_n^+ . We claim that x_k satisfies $a_2 \dots a_{k-1} \preceq_R x_k$ and $b_2 \dots b_{k-1} \preceq_R x_k$.

Given the claim, the proof of (b) will follow since $a = (a_2 \dots a_{p+1}) \preceq_R x$ and $b = (b_2 \dots b_{p+1}) \preceq_R x$ implies that $c_{p+1} \preceq_R x$, so $x = x_{p+1} c_{p+1}$. But then x_{p+1} satisfies $a_2 \dots a_p \preceq_R x_{p+1}$ and $b_2 \dots b_p \preceq_R x_{p+1}$ by the claim for $k = p+1$. In particular, $c_p \preceq_R x_{p+1}$ and it follows that $x = x_p c_p c_{p+1}$. Continuing in this manner, we arrive at $x = x_2 (c_2 \dots c_{p+1}) = x_2 c$ and so $c \preceq_R x$. It therefore remains to prove the claim.

Since $c_k \dots c_{p+1} = (a'_k a_k) \dots (a'_{p+1} a_{p+1}) = (a'_k \dots a'_{p+1})(a_k \dots a_{p+1})$ it follows that

$$x = x_k (c_k \dots c_{p+1})$$

$$\begin{aligned}
 &= x_k(a'_k \dots a'_{p+1})(a_k \dots a_{p+1}) \\
 &= y_k(a_k \dots a_{p+1}) \text{ for } y_k = x_k(a'_k \dots a'_{p+1}).
 \end{aligned}$$

Since x is a common multiple of a and b , then we also have $a = (a_2 \dots a_{p+1}) \preceq_R x$, that is, for some z_k .

$$x = z_k(a_2 \dots a_{p+1})$$

Therefore, by cancellation of $(a_k \dots a_{p+1})$,

$$y_k = z_k(a_2 \dots a_{k-1}).$$

By Lemma 4.27, $\Delta((a'_k \dots a'_{p+1}), (a_2 \dots a_{k-1})) \preceq_R y_k$. Since the two words letterwise commute $\Delta((a'_k \dots a'_{p+1}), (a_2 \dots a_{k-1})) = (a_2 \dots a_{k-1})(a'_k \dots a'_{p+1})$ and so

$$y_k = w_k(a_2 \dots a_{k-1})(a'_k \dots a'_{p+1})$$

for some w_k in A_n^+ . It follows

$$\begin{aligned}
 x &= x_k(c_k \dots c_{p+1}) \\
 &= y_k(a_k \dots a_{p+1}) \\
 &= w_k(a_2 \dots a_{k-1})(a'_k \dots a'_{p+1})(a_k \dots a_{p+1}) \\
 &= w_k(a_2 \dots a_{k-1})((a'_k \dots a'_{p+1})(a_k \dots a_{p+1})) \\
 &= w_k(a_2 \dots a_{k-1})(c_k \dots c_{p+1})
 \end{aligned}$$

and by cancellation of $c_k \dots c_{p+1}$ on the first and final lines of the above equation, $(a_2 \dots a_{k-1}) \preceq_R x_k$ as required. The proof for $(b_2 \dots b_{k-1}) \preceq_R x_k$ is identical. This completes the proof of the Claim and thus of (b). \square

Recall that we have fixed α in A_n^+ with $\ell(\alpha) = k + 1$, and we have fixed a face $[\alpha a_2 \dots a_{p+1}]_p$ in $\mathcal{C}^n(k)$. We want to show that one of the facets of $[\alpha]_0$ from Lemma 8.9 is also in $\mathcal{C}^n(k)$. Recall since $[\alpha a]_p$ is in $\mathcal{C}^n(k)$, there exists β in A_n^+ such that $\ell(\beta) \leq k$ and

$$[[\alpha a]_p = [\alpha a_2 \dots a_{p+1}]_p = [\beta b_2 \dots b_{p+1}]_p = [\beta b]_p$$

from Definition 8.12, where $\ell(a_j) = i_j$ and $\ell(b_j) = l_j$. We have assumed β and b_j are chosen such that $\sum_{k=2}^{p+1} l_k$ is minimal, so we have either $b = e$, or $\ell(\beta) = \ell(\alpha) - 1$. Recall $\gamma := \overline{\alpha a} = \overline{\beta b}$, and that we defined u and v in A_p^+ such that

$$\alpha a = \gamma u \text{ and } \beta b = \gamma v.$$

We prove in the next three Lemmas that in the case $\text{EndGen}_p(\alpha a) \neq \emptyset$ we are done.

LEMMA 8.19. *If $\text{EndGen}_0(\alpha a) \neq \emptyset$, then the facet $[\alpha a_2]_1$ containing $[\alpha a]_p$ is in $\mathcal{C}^n(k)$.*

Proof. Consider τ in $\text{EndGen}_0(\alpha a)$. Then since the generators S_0 of A_0^+ commute with $\sigma_2, \dots, \sigma_n$ it follows that τ letterwise commutes (Definition 4.26) with a , because $a = a_2 \dots a_{p+1}$ only contains generators in the set of $\{\sigma_2, \dots, \sigma_n\}$. Since τ and a are both in $\text{EndMon}_n(\alpha a)$ and they letterwise commute, it follows from Lemma 4.28 that τ is in $\text{EndMon}_n(\alpha)$, that is, some α' in A_n^+ with $\ell(\alpha') < \ell(\alpha)$ satisfies $\alpha = \alpha' \tau$.

The facet $[\alpha a_2]_1$ therefore satisfies

$$[[\alpha a_2]_1 = [\alpha' \tau a_2]_1 = [\alpha' a_2 \tau]_1 = [\alpha' a_2]_1.$$

Here, the final equality is due to $\overline{\alpha' a_2 \tau} = \overline{\alpha' a_2}$ where the reduction is taken with respect to A_1^+ (from Lemma 4.19). The penultimate equality is due to the fact τ and a_2 letterwise commute.

Since $\ell(\alpha') < \ell(\alpha)$, $[[\alpha']]_0$ is in $\mathcal{C}^n(k)$ and $[[\alpha'a_2]]_1$ is a facet of $[[\alpha']]_0$. Therefore, $[[\alpha a_2]]_1$ is in $\mathcal{C}^n(k)$ and this completes the proof. \square

The case where $\text{EndGen}_p(\alpha a) \neq \emptyset$ but $\text{EndGen}_0(\alpha a) = \emptyset$ requires the following technical lemma.

LEMMA 8.20. *Suppose $a_j \neq e$, then the words a_j and a_{j+1} as in Definition 8.7 satisfy $a_{j+1}\sigma_j = \bar{a}_j a_j$, for some \bar{a}_j in A_n^+ with $\ell(\bar{a}_j) \geq 1$. Furthermore, \bar{a}_j letterwise commutes with $a_2 \dots a_{j-1}$. Regardless of whether or not $a_j = e$, $a_{j+1}\sigma_j$ corresponds to the face map $\partial_{i_{j+1}+1}^{n-j+1}$. The analogous results hold for the b_j .*

Proof. If $a_j \neq e$, then

$$\begin{aligned} a_{j+1}\sigma_j &= (\sigma_{i_{j+1}+j} \dots \sigma_{j+1})\sigma_j \\ &= (\sigma_{i_{j+1}+j} \dots \sigma_{i_j+j})(\sigma_{i_j+j-1} \dots \sigma_{j+1})\sigma_j \\ &= (\sigma_{i_{j+1}+j} \dots \sigma_{i_j+j})(\sigma_{i_j+j-1} \dots \sigma_{j+1}\sigma_j) \\ &= (\sigma_{i_{j+1}+j} \dots \sigma_{i_j+j})a_j \\ &= \bar{a}_j a_j, \end{aligned}$$

so $\bar{a}_j = \sigma_{i_{j+1}+j} \dots \sigma_{i_j+j}$, and $\ell(\bar{a}_j) \geq 1$ since $\ell(a_{j+1}) \geq \ell(a_j) \geq 1$. The generators appearing in the word $a_2 \dots a_{j-1}$ are $\{\sigma_2, \dots, \sigma_{i_{j-1}+(j-1)-1}\}$ and so to prove that \bar{a}_j letterwise commutes with $a_2 \dots a_{j-1}$ it is enough to show that the sets $A = \{\sigma_{i_j+j}, \dots, \sigma_{i_{j+1}+j}\}$ and $B = \{\sigma_2, \dots, \sigma_{i_{j-1}+(j-1)-1}\}$ pairwise commute. The largest index of a generator in B is $i_{j-1} + (j-1) - 1$ and the smallest index of a generator in A is $i_j + j$ so it is enough to show

$$|(i_j + j) - (i_{j-1} + (j-1) - 1)| = |i_j - i_{j-1} + 2| \geq 2.$$

This holds since $i_j \geq i_{j-1}$, and so \bar{a}_j and $a_2 \dots a_{j-1}$ letterwise commute. Regardless of whether or not $a_j = e$, $a_{j+1}\sigma_j = \bar{a}_j a_j = \sigma_{i_{j+1}+j} \dots \sigma_j$ corresponds to the face map $\partial_{i_{j+1}+1}^{n-j+1}$ defined in Definition 8.7. Since the b_j have the same form as the a_j , with difference only in word length, the analogous statements hold for the b_j . \square

LEMMA 8.21. *If $\text{EndGen}_p(\alpha a) \neq \emptyset$ but $\text{EndGen}_0(\alpha a) = \emptyset$, then some σ_j is in $\text{EndGen}_p(\alpha a)$ for $1 \leq j \leq p$. Then the facet $[[\alpha a_j \sigma_{j-1} \dots \sigma_2]]_1$ containing $[[\alpha a]]_p$ is in $\mathcal{C}^n(k)$.*

Proof. If $\text{EndGen}_0(\alpha a) = \emptyset$ and $\text{EndGen}_p(\alpha a) \neq \emptyset$ it follows that

$$\{\sigma_1, \sigma_2, \dots, \sigma_p\} \cap \text{EndGen}_p(\alpha a) \neq \emptyset,$$

so some σ_j is in $\text{EndGen}_p(\alpha a)$ for $1 \leq j \leq p$. Then σ_j and $a = a_2 \dots a_{p+1}$ are both in $\text{EndMon}_n(\alpha a)$. In particular, σ_j and $a_{j+2} \dots a_{p+1}$ are both in $\text{EndMon}_n(\alpha a)$. Since σ_j and $a_{j+2} \dots a_{p+1}$ letterwise commute we have from Lemma 4.28 that σ_j is in $\text{EndMon}_n(\alpha a_2 \dots a_{j+1})$. Since a_{j+1} is also in $\text{EndMon}_n(\alpha a_2 \dots a_{j+1})$, from Lemma 4.25 $\Delta(a_{j+1}, \sigma_j)$ is in $\text{EndMon}_n(\alpha a_2 \dots a_{j+1})$. Also, from Lemma 8.15, $\Delta(a_{j+1}, \sigma_j) = a_{j+1}\sigma_j a_{j+1}$. By cancellation of a_{j+1} , it follows that $a_{j+1}\sigma_j$ is in $\text{EndMon}_n(\alpha a_2 \dots a_j)$, so

$$\alpha a_2 \dots a_j = \alpha'(a_{j+1}\sigma_j) \tag{\dagger}$$

for some α' in A_n^+ .

Recall Lemma 8.20 and split into two cases: either

- (a) $a_j \neq e$, or
- (b) $a_2 = \dots = a_j = e$ since $\ell(a_i) \leq \ell(a_{i+1}) \forall i$.

For case (a) recall from Lemma 8.20 that $a_{j+1}\sigma_j = \bar{a}_j a_j$ and \bar{a}_j letterwise commutes with $a_2 \dots a_{j-1}$. This gives

$$\begin{aligned} \alpha a_2 \dots a_j &= \alpha'(a_{j+1}\sigma_j) \text{ from Equation } (\dagger) \\ &= \alpha'(\bar{a}_j a_j) \\ \Rightarrow \alpha a_2 \dots a_{j-1} &= \alpha' \bar{a}_j \text{ by cancellation of } a_j. \end{aligned}$$

Now $\alpha(a_2 \dots a_{j-1}) = \alpha' \bar{a}_j$ and \bar{a}_j letterwise commutes with $a_2 \dots a_{j-1}$. By Lemma 4.28 it follows that \bar{a}_j is in $\text{EndMon}_n(\alpha)$, that is there exists α'' in A_n^+ such that $\alpha = \alpha'' \bar{a}_j$.

Then the facet $\llbracket \alpha a_j \sigma_{j-1} \dots \sigma_2 \rrbracket_1$ satisfies

$$\begin{aligned} &\llbracket \alpha a_j \sigma_{j-1} \dots \sigma_2 \rrbracket_1 \\ &= \llbracket \alpha'' \bar{a}_j a_j \sigma_{j-1} \dots \sigma_2 \rrbracket_1 \end{aligned}$$

and by Lemma 8.20, $\bar{a}_j a_j$ is a face map $\partial_{i_{j+1}+1}^{n-j+1}$, so $\bar{a}_j a_j \sigma_{j-1} \dots \sigma_2$ is also a face map $\partial_{i_{j+1}+j-1}^{n-1}$, and therefore $\llbracket \alpha a_j \sigma_{j-1} \dots \sigma_2 \rrbracket_1$ is also a facet of $\llbracket \alpha'' \rrbracket_0$. Since $\ell(\bar{a}_j) \geq 1$ by Lemma 8.20 it follows $\ell(\alpha'') < \ell(\alpha)$ and so $\llbracket \alpha a_j \sigma_{j-1} \dots \sigma_2 \rrbracket_1 \in \mathcal{C}^n(k)$.

For case (b), $a_2 = \dots = a_j = e$ implies $a_{j+1}\sigma_j$ is in $\text{EndMon}_n(\alpha)$, so $\alpha = \alpha' a_{j+1}\sigma_j$ for some α' in A_n^+ with $\ell(\alpha') < \ell(\alpha)$. Then the facet $\llbracket \alpha a_j \sigma_{j-1} \dots \sigma_2 \rrbracket_1$ satisfies

$$\begin{aligned} &\llbracket \alpha a_j \sigma_{j-1} \dots \sigma_2 \rrbracket_1 \\ &= \llbracket (\alpha' a_{j+1}\sigma_j) a_j \sigma_{j-1} \dots \sigma_2 \rrbracket_1 \\ &= \llbracket \alpha' (a_{j+1}\sigma_j \sigma_{j-1} \dots \sigma_2) \rrbracket_1 \text{ since } a_j = e \end{aligned}$$

and as before by Lemma 8.20, this is a face of $\llbracket \alpha' \rrbracket_0$ which is in $\mathcal{C}^n(k)$ as required. □

PROPOSITION 8.22. *If $\text{EndGen}_p(\alpha a) \neq \emptyset$ and $\llbracket \alpha a \rrbracket_p$ is in $\mathcal{C}^n(k)$, then a facet containing $\llbracket \alpha a \rrbracket_p$ is in $\mathcal{C}^n(k)$.*

Proof. Putting together Lemmas 8.19 and 8.21 gives the required result. □

8.5. *Proof of point (A): case (i): $\ell(\beta b) < \ell(\alpha a)$*

PROPOSITION 8.23. *Under the hypotheses of case (i), $\text{EndGen}_p(\alpha a) \neq \emptyset$.*

Proof. Recall that for some u and v in A_p^+ , $\alpha a = \gamma u$ and $\beta b = \gamma v$. If $\ell(\beta b) < \ell(\alpha a)$, then it follows $\ell(\gamma v) < \ell(\gamma u)$ and consequently $\ell(v) < \ell(u)$, since multiplication in A_n^+ corresponds to addition of lengths. Since the inequality is strict, it follows that $\ell(u) \neq 0$, that is, $u \neq e$. Then since $\alpha a = \gamma u$, $u \in \text{EndMon}_p(\alpha a)$ so in particular $\text{EndGen}_p(\alpha a) \neq \emptyset$. □

Applying Proposition 8.22 concludes the proof of case (i).

8.6. *Proof of point (A): case (ii): $\ell(\beta b) = \ell(\alpha a)$*

Recall that for some u and v in A_p^+ , and γ in A_n^+ with $\text{EndMon}_p(\gamma) = \emptyset$, that $\alpha a = \gamma u$ and $\beta b = \gamma v$.

PROPOSITION 8.24. *If we are in case (ii), then we only need to consider when $\alpha a = \beta b = \gamma$.*

Proof. Case (ii) states that $\ell(\beta b) = \ell(\alpha a)$. This implies that $\ell(\gamma u) = \ell(\gamma v)$, which in turn implies $\ell(u) = \ell(v)$ by cancellation. If $u \neq e$, then αa satisfies $\text{EndGen}_p(\alpha a) \neq \emptyset$. Then, by

Proposition 8.22 a facet containing $[[\alpha a]]_p$ lies in $\mathcal{C}^n(k)$. Therefore, we can assume $u = e$, which implies $v = e$ since they have the same length. Therefore, $\alpha a = \beta b = \gamma$. \square

Recall the definition of c_j from Definition 8.17:

$$c_j = \begin{cases} a_j & \text{if } \ell(a_j) \geq \ell(b_j) \\ b_j & \text{if } \ell(a_j) < \ell(b_j) \end{cases}$$

for $2 \leq j \leq p + 1$. Recall $c = c_2 \dots c_{p+1}$. Recall that since $\ell(\beta) < \ell(\alpha)$, then in case (ii): $\ell(\beta b) = \ell(\alpha a)$ that it follows $\ell(b) > \ell(a)$.

PROPOSITION 8.25. *With the notation as above, there exists at least one j for which $c_j = b_j \neq a_j$. Consider the maximal j for which $c_j = b_j \neq a_j$. Then the facet $[[\alpha a_j \sigma_{j-1} \dots \sigma_2]]_1$ of $[[\alpha]]_0$ containing $[[\alpha a]]_p$ is in $\mathcal{C}^n(k)$.*

Proof. Recall $c = a'a = b'b$, where $a' = a'_2 \dots a'_{p+1}$ and $b' = b'_2 \dots b'_{p+1}$ as in Definition 8.17. We first prove the existence of j in the statement. Note since $\ell(\beta) < \ell(\alpha)$, it follows that $b \neq e$ and so from Lemma 8.11, it follows that $\ell(\beta) = \ell(\alpha) - 1$ which gives $\ell(b) = \ell(a) + 1$. Since $c = a'a = b'b$, this gives $\ell(a') = \ell(b') + 1$ and in particular $\ell(a') \geq 1$. It follows that at least one $a'_j \neq e$, that is, $c_j = b_j \neq a_j$.

Recall also that $\alpha a = \beta b = \gamma$ from Proposition 8.24. Therefore, a and b are in $\text{EndMon}_n(\alpha a)$ and it follows from Lemma 4.25 that $\Delta(a, b)$ is in $\text{EndMon}_n(\alpha a)$. From Lemma 8.18 $\Delta(a, b) = c$ so it follows that c is in $\text{EndMon}_n(\alpha a)$, that is, for some a' in A_n^+ with $\ell(a') < \ell(\alpha)$

$$\alpha a = \alpha'(c) = \alpha'(a'a).$$

By cancellation of a , we have $\alpha = \alpha'a'$.

Consider the maximal j for which $c_j = b_j \neq a_j$. Then $a'_{j+1} = \dots = a'_{p+1} = e$, that is, $a' = a'_2 \dots a'_j$. It follows that the facet $[[\alpha a_j \sigma_{j-1} \dots \sigma_2]]_1$ satisfies

$$\begin{aligned} [[(\alpha) a_j \sigma_{j-1} \dots \sigma_2]]_1 &= [[(\alpha' a') a_j \sigma_{j-1} \dots \sigma_2]]_1 \\ &= [[(\alpha' a'_2 \dots a'_j) a_j \sigma_{j-1} \dots \sigma_2]]_1 \\ &= [[\alpha' a'_2 \dots (a'_j a_j) \sigma_{j-1} \dots \sigma_2]]_1 \\ &= [[\alpha' a'_2 \dots (c_j) \sigma_{j-1} \dots \sigma_2]]_1 \\ &= [[\alpha' a'_2 \dots a'_{j-1} (b_j) \sigma_{j-1} \dots \sigma_2]]_1. \end{aligned}$$

Recall $\ell(b_j) = l_j$, so post multiplication by $b_j \sigma_{j-1} \dots \sigma_2$ corresponds to the face map $\partial_{l_j+j-2}^{n-1}$. Therefore, $[[\alpha a_j \sigma_{j-1} \dots \sigma_2]]_1$ is a facet of $[[\alpha' a'_2 \dots a'_{j-1}]]_0$ and we have that $\ell(\alpha' a'_2 \dots a'_{j-1}) < \ell(\alpha)$ since $\alpha = \alpha' a'_2 \dots a'_j$ and $\ell(a'_j) \geq 1$. Therefore, $[[\alpha a_j \sigma_{j-1} \dots \sigma_2]]_1$ is in $\mathcal{C}^n(k)$. \square

8.7. Proof of point (A): case (iii): $\ell(\beta b) > \ell(\alpha a)$

Recall that for some u and v in A_p^+ , and γ in A_n^+ with $\text{EndMon}_p(\gamma) = \emptyset$, that $\alpha a = \gamma u$ and $\beta b = \gamma v$.

PROPOSITION 8.26. *If we are in case (iii), then $b \neq e$. Furthermore, we only need to consider the case when $\gamma = \alpha a$ so $\beta b = \gamma v = \alpha a v$. In this case it follows $\text{EndGen}_p(\beta b) \neq \emptyset$.*

Proof. Case (iii) states that $\ell(\beta b) > \ell(\alpha a)$, and note that this can only happen when $b \neq e$ since $\ell(\beta) < \ell(\alpha)$. Recall this implies $\ell(\beta) = \ell(\alpha) - 1$ from Lemma 8.11. If $u \neq e$, then αa satisfies $\text{EndGen}_p(\alpha a) \neq \emptyset$. Then by Proposition 8.22, a facet containing $[[\alpha a]]_p$ lies

in $\mathcal{C}^n(k)$. Therefore, we can assume $u = e$. Then $\alpha a = \gamma$ and it follows that $\beta b = \gamma v = \alpha a v$. Since $\ell(\beta b) > \ell(\alpha a)$ it follows $\ell(v) \geq 1$ and therefore $\text{EndGen}_p(\beta b) \neq \emptyset$. \square

We now prove a technical lemma required for the rest of this section.

LEMMA 8.27. *If there exists $\beta' \in A_n^+$ such that $\ell(\beta') = \ell(\beta) - 1 = \ell(\alpha) - 2$ and*

$$[[\beta b]]_p = [[\beta' b]]_p,$$

then this contradicts our choice of b .

Proof. Write $b = \sigma' b'$, that is, σ' is the leftmost generator of the word b . Then $\ell(\beta' \sigma') = \ell(\alpha) - 1$ and so

$$[[\beta' \sigma' b']]_p = [[\beta b]]_p,$$

where $\ell(b') < \ell(b)$ and $\ell(\beta' \sigma') = \ell(\beta) = \ell(\alpha) - 1$. This contradicts our choice of b : we chose b such that $\sum_{k=2}^{p+1} l_k$ was minimal, as in Lemma 8.11 and therefore no such b' can exist. \square

PROPOSITION 8.28. $\text{EndGen}_0(\beta b) = \emptyset$.

Proof. Suppose $\text{EndGen}_0(\beta b) \neq \emptyset$ for a contraction. Let τ in $\text{EndGen}_0(\beta b)$. Then since τ letterwise commutes with $b_2 \dots b_{p+1}$ it follows that τ is in $\text{EndGen}_0(\beta)$ from Lemma 4.28. Then $\beta = \beta' \tau$ for some β' in A_n^+ with $\ell(\beta') = \ell(\beta) - 1 = \ell(\alpha) - 2$. It follows

$$\begin{aligned} [[\beta b]]_p &= [[(\beta' \tau) b]]_p \\ &= [[\beta' \tau b]]_p \\ &= [[\beta' b \tau]]_p \\ &= [[\beta' b]]_p \end{aligned}$$

and by Lemma 8.27, this is a contradiction. \square

PROPOSITION 8.29. *The generator σ_1 is not in $\text{EndGen}_p(\beta b)$.*

Proof. Suppose σ_1 is in $\text{EndGen}_p(\beta b) \neq \emptyset$ and work for a contradiction. Since σ_1 letterwise commutes with $b_3 \dots b_{p+1}$ it follows that σ_1 is in $\text{EndGen}_p(\beta b_2)$ by Lemma 4.28. From Lemma 8.15, $\Delta(\sigma_1, b_2) = b_2 \sigma_1 b_2$ and by Lemma 4.27, this is in $\text{EndMon}_n(\beta b_2)$, giving by cancellation of b_2 that $b_2 \sigma_1$ is in $\text{EndMon}_n(\beta)$. So $\beta = \beta' b_2 \sigma_1$ for some β' in A_n^+ . Then

$$\begin{aligned} [[(\beta)(b)]_p &= [[(\beta' b_2 \sigma_1)(b)]_p \\ &= [[(\beta' b_2 \sigma_1)(b_2 \dots b_{p+1})]]_p \end{aligned}$$

and by Lemma 8.16, $b_2 \sigma_1 b_2$ can be written as $\hat{b}_1 \sigma_1 b_2 \sigma_1$. So, we have

$$\begin{aligned} [[(\beta)(b)]_p &= [[(\beta' b_2 \sigma_1)(b_2 \dots b_{p+1})]]_p \\ &= [[\beta'(b_2 \sigma_1 b_2)(b_3 \dots b_{p+1})]]_p \\ &= [[\beta'(\hat{b}_1 \sigma_1 b_2 \sigma_1)(b_3 \dots b_{p+1})]]_p \\ &= [[\beta'(\hat{b}_1 \sigma_1 b_2)(b_3 \dots b_{p+1}) \sigma_1]]_p \\ &= [[\beta'(\hat{b}_1 \sigma_1)(b_2 b_3 \dots b_{p+1}) \sigma_1]]_p \end{aligned}$$

$$\begin{aligned}
&= \llbracket \beta' \hat{b}_1 \sigma_1(b) \sigma_1 \rrbracket_p \\
&= \llbracket \beta' \hat{b}_1 \sigma_1 b \rrbracket_p
\end{aligned}$$

with $\ell(\beta' \hat{b} \sigma_1) = \ell(\beta) - 1$, since $\beta = \beta' b_2 \sigma_1$ and $\hat{b}_1 = \sigma_{i_2} \dots \sigma_2$ is a subword of $b_2 = \sigma_{i_2+1} \dots \sigma_2$ satisfying $\ell(\hat{b}) = \ell(b) - 1$. By Lemma 8.27, this is a contradiction. \square

PROPOSITION 8.30. *The generator σ_j is not in $\text{EndGen}_p(\beta b)$ for $2 \leq j \leq p$.*

Proof. Suppose σ_j is in $\text{EndGen}_p(\beta b) = \text{EndGen}_p(\beta(b_2 \dots b_{p+1}))$ for some $2 \leq j \leq p$ and work for a contradiction. Since σ_j letterwise commutes with $b_{j+2} \dots b_{p+1}$ it follows from Lemma 4.28 that σ_j is in $\text{EndGen}_p(\beta b_2 \dots b_{j+1})$. From Lemma 8.15, $\Delta(\sigma_j, b_{j+1}) = b_{j+1} \sigma_j b_{j+1}$ and by Lemma 4.27 this is in $\text{EndMon}_n(\beta b_2 \dots b_{j+1})$, giving by cancellation of b_{j+1} that $b_{j+1} \sigma_j$ is in $\text{EndMon}_n(\beta b_2 \dots b_j)$. We first handle the two cases, where $b_j = e$, namely the case, where $b_{j+1} = e$ and the case, where $b_{j+1} \neq e$.

When $b_{j+1} = e$, it follows from the conditions on the l_i that $b_j = b_{j-1} = \dots = b_2 = e$, so σ_j is in $\text{EndMon}_n(\beta)$, that is, there exists $\beta' \in A_n^+$ such that $\beta = \beta' \sigma_j$ (in particular, $\ell(\beta') = \ell(\beta) - 1$). In this case

$$\begin{aligned}
\llbracket [\beta b] \rrbracket_p &= \llbracket \beta' \sigma_j b_{j+2} \dots b_{p+1} \rrbracket_p \\
&= \llbracket \beta' b_{j+2} \dots b_{p+1} \sigma_j \rrbracket_p \\
&= \llbracket \beta' b_{j+2} \dots b_{p+1} \rrbracket_p \\
&= \llbracket \beta' b \rrbracket_p
\end{aligned}$$

and by Lemma 8.27 this is a contradiction.

When $b_j = e$ but $b_{j+1} \neq e$, it follows that $b_{j-1} = \dots = b_2 = e$, so $b_{j+1} \sigma_j$ is in $\text{EndMon}_n(\beta)$, that is, there exists $\beta' \in A_n^+$ such that $\beta = \beta' b_{j+1} \sigma_j$ and therefore

$$\begin{aligned}
\llbracket [\beta b] \rrbracket_p &= \llbracket \beta' (b_{j+1} \sigma_j b_{j+1}) \dots b_{p+1} \rrbracket_p \\
&= \llbracket \beta' (\hat{b}_j b_j b_{j+1} \sigma_j) b_{j+2} \dots b_{p+1} \rrbracket_p \text{ by Lemma 8.16} \\
&= \llbracket \beta' \hat{b}_j b_{j+1} \sigma_j b_{j+2} \dots b_{p+1} \rrbracket_p \text{ since } b_j = e \\
&= \llbracket \beta' \hat{b}_j b_{j+1} b_{j+2} \dots b_{p+1} \sigma_j \rrbracket_p \\
&= \llbracket \beta' \hat{b}_j b \rrbracket_p.
\end{aligned}$$

We note that in this case, since $b_j = e$, $\hat{b}_j = \sigma_{l_{j+1}+j-1} \dots \sigma_j$ and this has the same length as $b_{j+1} = \sigma_{l_{j+1}+j} \dots \sigma_{j+1}$. Therefore, $\ell(\beta' \hat{b}_j) = \ell(\beta' b_{j+1}) = \ell(\beta) - 1$ and by Lemma 8.27 this is again a contradiction.

Now assume $b_j \neq e$. By Lemma 8.20, $b_{j+1} \sigma_j = \bar{b}_j b_j$ with $\ell(\bar{b}_j) \geq 1$, and so by cancellation of b_j , \bar{b}_j is in $\text{EndMon}_n(\beta b_2 \dots b_{j-1})$. From Lemma 4.28, since \bar{b}_j letterwise commutes with $b_2 \dots b_{j-1}$ we have \bar{b}_j is in $\text{EndMon}_n(\beta)$ so $\beta = \beta' \bar{b}_j$ for some β' in A_n^+ . Then it follows that

$$\begin{aligned}
\llbracket [(\beta) b] \rrbracket_p &= \llbracket (\beta' \bar{b}_j)(b) \rrbracket_p \\
&= \llbracket (\beta' \bar{b}_j)(b_2 \dots b_{p+1}) \rrbracket_p \\
&= \llbracket (\beta' \bar{b}_j)(b_2 \dots b_{j-1}) b_j (b_{j+1} \dots b_{p+1}) \rrbracket_p \\
&= \llbracket (\beta')(b_2 \dots b_{j-1}) (\bar{b}_j b_j) (b_{j+1} \dots b_{p+1}) \rrbracket_p \\
&= \llbracket \beta' (b_2 \dots b_{j-1}) (b_{j+1} \sigma_j) (b_{j+1} \dots b_{p+1}) \rrbracket_p \text{ since } \bar{b}_j b_j = b_{j+1} \sigma_j
\end{aligned}$$

$$\begin{aligned}
 &= \llbracket \beta'(b_2 \dots b_{j-1})(b_{j+1}\sigma_j b_{j+1})(b_{j+2} \dots b_{p+1}) \rrbracket_p \\
 &= \llbracket \beta'(b_2 \dots b_{j-1})(\hat{b}_j b_j b_{j+1} \sigma_j)(b_{j+2} \dots b_{p+1}) \rrbracket_p \text{ by Lemma 8.16} \\
 &= \llbracket \beta'(b_2 \dots b_{j-1})(\hat{b}_j)(b_j b_{j+1})(\sigma_j)(b_{j+2} \dots b_{p+1}) \rrbracket_p \\
 &= \llbracket \beta' \hat{b}_j (b_2 \dots b_{j-1})(b_j b_{j+1}) \sigma_j (b_{j+2} \dots b_{p+1}) \rrbracket_p \\
 &= \llbracket \beta' \hat{b}_j (b_2 \dots b_{j-1} b_j b_{j+1} b_{j+2} \dots b_{p+1}) \sigma_j \rrbracket_p \\
 &= \llbracket \beta' \hat{b}_j (b) \sigma_j \rrbracket_p \\
 &= \llbracket \beta' \hat{b}_j b \rrbracket_p.
 \end{aligned}$$

Since $\beta b = \beta' \hat{b}_j b \sigma_j$, it follows from the additive property of the length function that $\ell(\beta' \hat{b}_j) = \ell(\beta) - 1$ and so by Lemma 8.27 this is a contradiction. \square

By Propositions 8.28, 8.29 and 8.30, it follows that $\text{EndGen}_p(\beta b) = \emptyset$. This contradicts the statement of Proposition 8.26 and therefore concludes the proof of case (iii) and hence the proof of point (A).

8.8. Proof of point (B)

Recall point B: If $\ell(\alpha) = \ell(\beta) = k + 1$ and $\alpha \neq \beta$, then $\llbracket \alpha \rrbracket_0 \cap \llbracket \beta \rrbracket_0 \subseteq \mathcal{C}^n(k)$.

PROPOSITION 8.31. Suppose $\alpha \neq \beta$ in A_n^+ . If $\ell(\alpha) = \ell(\beta) = k + 1$, then it follows that $\llbracket \alpha \rrbracket_0 \cap \llbracket \beta \rrbracket_0 \subseteq \mathcal{C}^n(k)$.

Proof. Suppose $\llbracket \alpha \rrbracket_0 \cap \llbracket \beta \rrbracket_0 \neq \emptyset$. Then for some $1 \leq p \leq n - 1$ there exists a and b as in Definition 8.12 such that $\llbracket \alpha a \rrbracket_p = \llbracket \beta b \rrbracket_p$. It follows that there exists γ in A_n^+ and u, v in A_p^+ such that

$$\alpha a = \gamma u \text{ and } \beta b = \gamma v.$$

Suppose that $u \neq e$. Then by Proposition 8.22, it follows that a facet of $\llbracket \alpha \rrbracket_0$ containing $\llbracket \alpha a \rrbracket_p$ is in $\mathcal{C}^n(k)$. Hence, $\llbracket \alpha a \rrbracket_p = \llbracket \beta b \rrbracket_p$ itself is in $\mathcal{C}^n(k)$. Similarly, if $v \neq e$, then a facet of $\llbracket \beta \rrbracket_0$ containing $\llbracket \beta b \rrbracket_p = \llbracket \alpha a \rrbracket_p$ is in $\mathcal{C}^n(k)$, and hence $\llbracket \beta b \rrbracket_p = \llbracket \alpha a \rrbracket_p$ itself is in $\mathcal{C}^n(k)$. So, we are left with the case that $u = v = e$, giving

$$\alpha a = \gamma = \beta b$$

and since $\ell(\alpha) = \ell(\beta)$ it follows that $\ell(a) = \ell(b)$. Since $\alpha \neq \beta$ it follows $a \neq b$. Recall the definition of c, a' and b' from Definition 8.17. From Lemma 8.18, $c = \Delta(a, b)$ and $c = a'a = b'b$. Since $\ell(a) = \ell(b)$, then $\ell(a') = \ell(b')$. Suppose $a' = e$, then $\ell(a') = \ell(b')$ gives $b' = e$ and hence $c = a = b$. But $a \neq b$ so it follows that $a' \neq e$ and in particular $\ell(a') \geq 1$.

From Lemma 4.27, since a and b are in $\text{EndMon}_n(\alpha a)$ it follows that $\Delta(a, b) = c$ is in $\text{EndMon}_n(\alpha a)$, so $\alpha a = \alpha' c = \alpha'(a'a)$ for some α' in A_n^+ . By cancellation of a we have $\alpha = \alpha'a'$ and $\ell(\alpha') < \ell(\alpha)$. Then

$$\begin{aligned}
 \llbracket \alpha a \rrbracket_p &= \llbracket (\alpha'a')a \rrbracket_p \\
 &= \llbracket \alpha' c \rrbracket_p
 \end{aligned}$$

and $\llbracket \alpha' c \rrbracket_p$ is in $\mathcal{C}^n(k)$ since c represents a series of face maps originating at $\llbracket \alpha' \rrbracket_0$, with each face map given by the map corresponding to right multiplication by c_j , which is either the face map corresponding to a_j or b_j . \square

This completes the proof of point (B), and hence by Proposition 8.4, it follows that $\|\mathcal{C}_\bullet^n\|$ is $(n - 2)$ connected.

9. Proof of Theorem A

This section proves the required results on the differentials of the spectral sequence introduced in Subsection 1.1, before putting together the results of the previous sections and running the spectral sequence argument to complete the proof of Theorem A.

9.1. Results on face and stabilisation maps

Recall the definition of the face maps of \mathcal{A}_\bullet^n from Definition 7.5:

$$\partial_k^p : \mathcal{A}_p^n \rightarrow \mathcal{A}_{p-1}^n \text{ for } 0 \leq k \leq p$$

and given by

$$\begin{aligned} \partial_k^p : \mathcal{A}_p^n &\rightarrow \mathcal{A}_{p-1}^n \\ \partial_k^p : A_n^+ \parallel \mathcal{C}_p^n &\rightarrow A_n^+ \parallel \mathcal{C}_{p-1}^n, \end{aligned}$$

where ∂_k^p is induced by the face maps of \mathcal{C}_\bullet^n , which we recall are a composite of right multiplication of the representative for the equivalence class in $\mathcal{C}_p^n = A^+(n; n - p - 1)$ by $(\sigma_{n-p+k}\sigma_{n-p+k-1} \dots \sigma_{n-p+1})$, before the inclusion to the equivalence class in \mathcal{C}_{p-1}^n .

Recall from Lemma 7.4 that for each $0 \leq p \leq n - 1$ there is a homotopy equivalence

$$A_n^+ \parallel A_{n-p-1}^+ \simeq A^+(n; n - p - 1) = \mathcal{C}_p^n,$$

given by the map defined levelwise on the bar construction by

$$\begin{aligned} B_k(A_n^+, A_{n-p-1}^+, *) &\rightarrow A^+(n; n - p - 1) \\ \alpha[m_1, \dots, m_k] &\mapsto \bar{\alpha}, \end{aligned}$$

where $\alpha \in A_n^+$, $m_i \in A_{n-p-1}^+$ for all i and $\alpha = \bar{\alpha}\beta$ for $\bar{\alpha} \in A^+(n; n - p - 1)$ and $\beta \in A_{n-p-1}^+$.

DEFINITION 9.1. Define the map

$$d_k^p : A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+ \rightarrow A_n^+ \parallel A_n^+ \parallel A_{n-p}^+$$

as the composition of two maps $\iota_p \circ \bar{d}_k^p$. The first map

$$\bar{d}_k^p : A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+ \rightarrow A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+$$

is given by right multiplication of the central term in the double homotopy quotient by $(\sigma_{n-p+k}\sigma_{n-p+k-1} \dots \sigma_{n-p+1})$.

The set of (j, k) -simplices in $A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+$ is identified with the product $(A_n^+)^j \times A_n^+ \times (A_{n-p-1}^+)^k$ and a generic element is given by $[a_1, \dots, a_j]a[a'_1, \dots, a'_k]$, where a_i and a are in A_n^+ and a'_i are in A_{n-p-1}^+ . The map \bar{d}_k^p acts on this simplex as

$$\bar{d}_k^p([a_1, \dots, a_j]a[a'_1, \dots, a'_k]) = [a_1, \dots, a_j]a(\sigma_{n-p+k}\sigma_{n-p+k-1} \dots \sigma_{n-p+1})[a'_1, \dots, a'_k]$$

The second map ι_p is the map

$$\iota_p : A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+ \rightarrow A_n^+ \parallel A_n^+ \parallel A_{n-p}^+$$

induced by the inclusion $A_{n-p-1}^+ \hookrightarrow A_{n-p}^+$. Note that \bar{d}_0^p is the identity map, and therefore $d_0^p = \iota_p$.

LEMMA 9.2. *The map \bar{d}_k^p in Definition 9.1 gives a well-defined map on the double homotopy quotient $A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+$.*

Proof. Since $(\sigma_{n-p+k}\sigma_{n-p+k-1}\dots\sigma_{n-p+1})$ letterwise commutes with every word in A_{n-p-1}^+ , it follows that \bar{d}_k^p commutes with all face maps of the bi-semi-simplicial space $A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+$. Therefore, the map on the central term of each simplex gives a map on the whole bi-semi-simplicial space, and hence its geometric realisation: the double homotopy quotient $A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+$. \square

LEMMA 9.3. *The diagram*

$$\begin{array}{ccc} A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+ & \xrightarrow{\cong} & A_n^+ \parallel \mathcal{C}_p^n \\ \downarrow d_k^p & & \downarrow \partial_k^p \\ A_n^+ \parallel A_n^+ \parallel A_{n-p}^+ & \xrightarrow{\cong} & A_n^+ \parallel \mathcal{C}_{p-1}^n \end{array}$$

commutes for all $p \geq 0$.

Proof. Recall from Lemma 7.8 that the horizontal homotopy equivalence is given by the levelwise maps on (j, k) -simplices of $A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+$:

$$\begin{aligned} (A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+)_{(j,k)} &\rightarrow (A_n^+ \parallel \mathcal{C}_p^n)_j \\ [a_1, \dots, a_j] \alpha [a'_1, \dots, a'_k] &\mapsto [a_1, \dots, a_j] \bar{\alpha}, \end{aligned}$$

where α and a_i are in A_n^+ , the a'_i are in A_{n-p-1}^+ , and $\alpha = \bar{\alpha}\beta$ for $\bar{\alpha}$ in $A^+(n; n-p-1)$ and β in A_{n-p-1}^+ . Diagram chasing using the definition of \bar{d}_k^p in Definition 9.1 gives that levelwise these maps commute, and so taking homotopy quotients and the corresponding maps induced by these levelwise maps yields the required result. \square

LEMMA 9.4. *The face maps ∂_k^p of A_\bullet^n are all homotopic to the zeroth face map ∂_0^p .*

Proof. The map \bar{d}_k^p restricted to $A_n^+ \parallel A_n^+$ is A_{n-p-1}^+ -equivariant, and the same holds for the identity map $\text{id}_{A_n^+ \parallel A_n^+}$. Applying Proposition 5.19 to these two maps therefore gives an A_{n-p-1}^+ -equivariant homotopy between them. It follows that they induce homotopic maps \bar{d}_k^p and $\text{id}_{A_n^+ \parallel A_n^+ / A_{n-p-1}^+}$ on $A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+$. Applying the inclusion ι_p to both maps and the homotopy between them yields a homotopy h_k from d_k^p to ι_p . However, ι_p is precisely the map d_0^p , and thus h_k is a homotopy from d_k^p to d_0^p for all k . Then the image of h_k under the homotopy equivalence in Lemma 9.3 yields a homotopy from ∂_k^p to the zeroth face map ∂_0^p , as required. \square

$$\begin{array}{ccc} A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+ & \xrightarrow{\cong} & A_n^+ \parallel \mathcal{C}_p^n \\ \downarrow \begin{pmatrix} d_k^p \\ \cong \\ d_0^p \end{pmatrix} & & \downarrow \begin{pmatrix} \partial_k^p \\ \cong \\ \partial_0^p \end{pmatrix} \\ A_n^+ \parallel A_n^+ \parallel A_{n-p}^+ & \xrightarrow{\cong} & A_n^+ \parallel \mathcal{C}_{p-1}^n. \end{array}$$

\square

LEMMA 9.5. *The following diagram commutes up to homotopy.*

$$\begin{array}{ccccc}
 BA_{n-p-1} & \xleftarrow{\simeq} & A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+ & \xrightarrow{\simeq} & A_n^+ \parallel \mathcal{C}_p^n \\
 \downarrow s_* & & d_k^p \left(\simeq \right) d_0^p & & \partial_k^p \left(\simeq \right) \partial_0^p \\
 BA_{n-p} & \xleftarrow{\simeq} & A_n^+ \parallel A_n^+ \parallel A_{n-p}^+ & \xrightarrow{\simeq} & A_n^+ \parallel \mathcal{C}_{p-1}^n
 \end{array}$$

that is, under the homotopy equivalence $\mathcal{A}_p^n \simeq BA_{n-p-1}^+$ of Lemma 7.8, the zeroth face map $\partial_0^p : \mathcal{A}_p^n \rightarrow \mathcal{A}_{p-1}^n$ is homotopy equivalent to the map $s_* : BA_{n-p-1}^+ \rightarrow BA_{n-p}^+$ induced by the stabilisation map $s : A_{n-p-1}^+ \hookrightarrow A_{n-p}^+$.

Proof. From the proof of Lemma 9.4, the right-hand square commutes up to homotopy.

From Lemma 7.8, the map from the centre to the left is given on the (j, k) -simplices of the geometric realisation by

$$\begin{aligned}
 f_{(j,k)}^p &: (A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+)_{(j,k)} \rightarrow (* \parallel A_{n-p-1}^+)_k \\
 &[a_1, \dots, a_j]a[a'_1, \dots, a'_k] \mapsto *[a'_1, \dots, a'_k],
 \end{aligned}$$

where a and a_i are in A_n^+ and a'_i is in A_{n-p-1}^+ . The map d_0^p is the map

$$d_0^p : A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+ \rightarrow A_n^+ \parallel A_n^+ \parallel A_{n-p}^+$$

induced by the inclusion $A_{n-p-1}^+ \hookrightarrow A_{n-p}^+$. Restricting this map to (j, k) -simplices of the double homotopy quotient gives

$$\begin{aligned}
 (d_0^p)_{(j,k)} &: (A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+)_{(j,k)} \rightarrow (A_n^+ \parallel A_n^+ \parallel A_{n-p}^+)_{(j,k)} \\
 &[a_1, \dots, a_j]a[a'_1, \dots, a'_k] \mapsto [a_1, \dots, a_j]a[a'_1, \dots, a'_k],
 \end{aligned}$$

where a and a_i are in A_n^+ and the a'_i are in A_{n-p-1}^+ , hence a'_i is in A_{n-p}^+ . Applying this map before the homotopy equivalence to the classifying space gives

$$\begin{array}{ccc}
 (A_n^+ \parallel A_n^+ \parallel A_{n-p-1}^+)_{(j,k)} & \xrightarrow{(d_0^p)_{(j,k)}} & (A_n^+ \parallel A_n^+ \parallel A_{n-p}^+)_{(j,k)} \\
 \downarrow f_{(j,k)}^p & & \downarrow f_{(j,k)}^{p-1} \\
 (* \parallel A_{n-p-1}^+)_k & & (* \parallel A_{n-p}^+)_k \\
 \parallel & & \parallel \\
 (BA_{n-p-1}^+)_k & \xrightarrow{\quad} & (BA_{n-p}^+)_k
 \end{array}$$

and on a (j, k) simplex this map is given by

$$\begin{array}{ccc}
 [a_1, \dots, a_j]a[a'_1, \dots, a'_k] & \xrightarrow{(d_0^p)_{(j,k)}} & [a_1, \dots, a_j]a[a'_1, \dots, a'_k] \\
 \downarrow f_{(j,k)}^p & & \downarrow f_{(j,k)}^{p-1} \\
 *[a'_1, \dots, a'_k] & \xrightarrow{\quad} & *[a'_1, \dots, a'_k].
 \end{array}$$

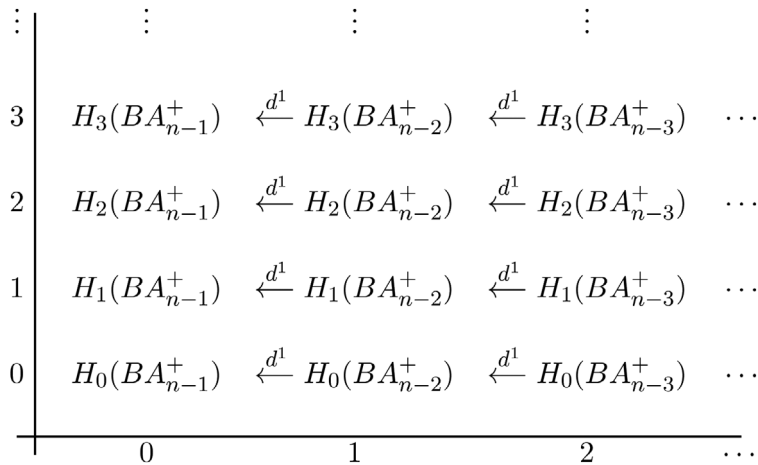


FIGURE 1. The E^1 page of the spectral sequence, with the groups identified.

We note that the dotted map is precisely the map which defines the natural inclusion $BA_{n-p-1}^+ \rightarrow BA_{n-p}^+$ under the identification of $*$ // A_r^+ with BA_r^+ for all r . The natural inclusion is in turn induced by the stabilisation map $A_r^+ \xrightarrow{s} A_{r+1}^+$ and so we denote it s_* . This gives that the left-hand square commutes up to homotopy. \square

9.2. Spectral sequence argument

In this section, we run a first quadrant spectral sequence for filtration of $\|\mathcal{A}_\bullet^n\|$, see, for example, Randal-Williams [20, 2 (sSS)]. Recall the points we proved regarding $\|\mathcal{A}_\bullet^n\|$:

- (1) there exist homotopy equivalences $\mathcal{A}_p^n \simeq BA_{n-p-1}^+$ for $p \geq 0$;
- (2) there is an $(n - 1)$ connected map $\|\phi_\bullet\|$ from the geometric realisation of \mathcal{A}_\bullet^n to the classifying space BA_n^+ :

$$\|\mathcal{A}_\bullet^n\| \xrightarrow{\|\phi_\bullet\|} BA_n^+,$$

that is, $\|\phi_\bullet\|$ induces an isomorphism on homotopy groups π_r for $0 \leq r \leq (n - 2)$, and a surjection for $r = (n - 1)$.

The first quadrant spectral sequence of the simplicial filtration of $\|\mathcal{A}_\bullet^n\|$ satisfies

$$E_{k,l}^1 = H_l(\mathcal{A}_k^n) \Rightarrow H_{k+l}(\|\mathcal{A}_\bullet^n\|).$$

By point (1), the left-hand side is given by $E_{k,l}^1 = H_l(\mathcal{A}_k^n) = H_l(BA_{n-k-1}^+)$. The first page of the spectral sequence is depicted in Figure 1. By point (2), $\|\phi_\bullet\|$ induces an isomorphism

$$H_{k+l}(\|\mathcal{A}_\bullet^n\|) \cong H_{k+l}(BA_n^+) \text{ when } (k + l) < n - 1$$

and a surjection

$$H_{k+l}(\|\mathcal{A}_\bullet^n\|) \twoheadrightarrow H_{k+l}(BA_n^+) \text{ when } (k + l) = n - 1.$$

The differential d^1 is given by an alternating sum of face maps in \mathcal{A}_\bullet^n . By Corollary 9.4, the face maps are all homotopic to each other and by Lemma 9.5 they are all homotopic to the stabilisation map s_* , via $\mathcal{A}_p^n \simeq BA_{n-p-1}^+$. Therefore, the alternating sum of face maps in the differential d^1 will cancel out to give the zero map when there are an even number of terms, and will give the stabilisation map when there are an odd number of terms, that is,

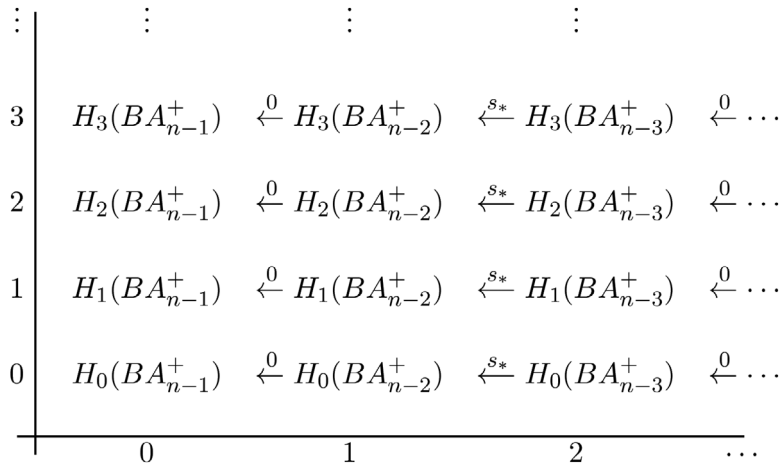


FIGURE 2. The E^1 page of the spectral sequence, with groups and d^1 differentials identified.

$d^1 : E_{\text{even},l}^1 \rightarrow E_{\text{odd},l}^1$ odd number of terms, so equals the stabilisation map s_* ,

$d^1 : E_{\text{odd},l}^1 \rightarrow E_{\text{even},l}^1$ even number of terms, so equals the zero map 0.

This gives the E^1 page shown in Figure 2.

We proceed by induction on n , for the sequence of monoids A_n^+ , and assume that homological stability holds for previous groups in the sequence. **Inductive hypothesis:** The map induced on homology by the stabilisation map

$$H_i(BA_{k-1}^+) \xrightarrow{s_*} H_i(BA_k^+)$$

is an isomorphism for $k > 2i$ and is a surjection for $k = 2i$ whenever $k < n$.

Here, we note that Theorem A holds for the base case $n = 1$, since we have to check $H_0(BA_0^+) \rightarrow H_0(BA_1^+)$ is a surjection, which is true since BA_n^+ is connected for all n .

LEMMA 9.6. *Under the inductive hypothesis, the $E_{0,l}$ terms stabilise on the E^1 page for $2l < n$, that is,*

$$E_{0,l}^1 = E_{0,l}^\infty \text{ when } 2l < n.$$

In particular, the d^1 differential does not alter these groups, and all possible sources of differentials mapping to $E_{0,l}$ for $2l < n$ are trivial from the E^2 page.

Proof. The d^1 differentials are given by either the zero map or the stabilisation map as shown in Figure 2. The d^1 differentials

$$d^1 : E_{1,l}^1 \rightarrow E_{0,l}^1$$

are given by the zero map, and the $E_{-1,l}^1$ terms are zero, since this is a first quadrant spectral sequence. Therefore, the $E_{0,l}^2$ terms are equal to the $E_{0,l}^1$ terms.

To show that the sources of all other differentials to $E_{0,l}$ for $2l < n$ are zero, we invoke the inductive hypothesis. This implies that on the E^1 page in the interior of the triangle of height $\lfloor \frac{n}{2} \rfloor$ and base n , the stabilisation maps, or d^1 differentials satisfy the inductive hypothesis. The resulting maps are shown in Figure 3, for the cases n odd and n even. Since the d^1 differentials going from the odd to the even columns are zero it follows that many groups in the interior of the triangle are zero on the E^2 page. This is shown in detail in Figure 4 for the

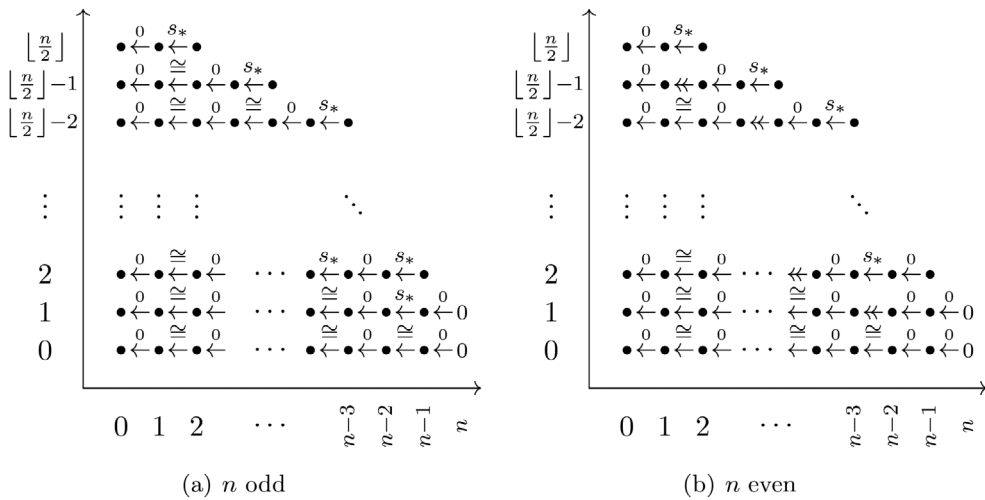


FIGURE 3. The E^1 page of the spectral sequence, with possible non-zero groups represented as circles and the inductive hypothesis applied to the d^1 differentials.

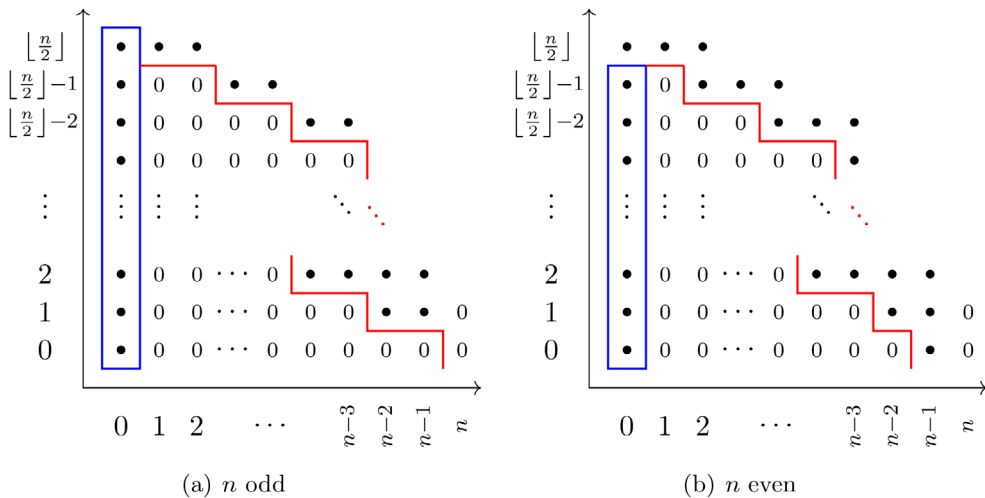


FIGURE 4 (colour online). The E^2 page of the spectral sequence, under the inductive hypothesis. To the left of the red line, all groups are zero except at positions $E_{0,l}^2$ for $2l < n$ — these are highlighted in blue.

cases n odd and n even. These groups include all the sources of differentials to $E_{0,l}$ for $2l < n$, hence $E_{0,l}^2 = E_{0,l}^\infty$ for $2l < n$. \square

We are now in a position to prove Theorem A.

THEOREM 9.7. *The sequence of monoids A_n^+ satisfies homological stability, that is*

$$H_i(BA_{n-1}^+) \cong H_i(BA_n^+)$$

when $2i < n$, and the map $H_i(BA_{n-1}^+) \rightarrow H_i(BA_n^+)$ is surjective when $2i = n$.

Proof. From Lemma 9.6, the spectral sequence satisfies

$$E_{0,i}^\infty = E_{0,i}^1 = H_i(BA_{n-1}^+)$$

when $2i < n$. From Proposition 7.11 and Theorem 8.1,

$$H_i(\|\mathcal{A}_\bullet^n\|) \cong H_i(BA_n^+)$$

when $i \leq n-2$, and the map $H_i(\|\mathcal{A}_\bullet^n\|) \rightarrow H_i(BA_n^+)$ is onto when $i = n-1$. The spectral sequence abuts to $H_{k+l}(\|\mathcal{A}_\bullet^n\|)$ and from Figure 4 the only non-zero groups on the diagonal $E_{k,l}^\infty$ when $k+l = i$ and $2i < n$ are the groups $E_{0,i}^\infty$. Putting these results together yields

$$H_i(BA_{n-1}^+) = E_{0,i}^\infty = H_{i+0}(\|\mathcal{A}_\bullet^n\|) \cong H_i(BA_n^+)$$

when both $i < \frac{n}{2}$ and $i \leq n-2$ are satisfied. When $n \geq 2$, $i < \frac{n}{2}$ implies $i \leq n-2$ and the case $n = 1$ was the base case of the inductive hypothesis. Therefore, an isomorphism is induced when $2i < n$.

When $i \leq n-1$ and $i < \frac{n}{2}$, it follows that

$$H_i(BA_{n-1}^+) = E_{0,i}^\infty = H_{i+0}(\|\mathcal{A}_\bullet^n\|) \twoheadrightarrow H_i(BA_n^+)$$

and for $n \geq 2$, $i < \frac{n}{2}$ implies $i \leq n-1$. Again the case $n = 1$ was the base case of the inductive hypothesis. This gives the required range for the surjection, and hence completes the proof. \square

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